Scaling relations of earthquakes, aseismic deformation and evolving fault structures in a damage rheology model

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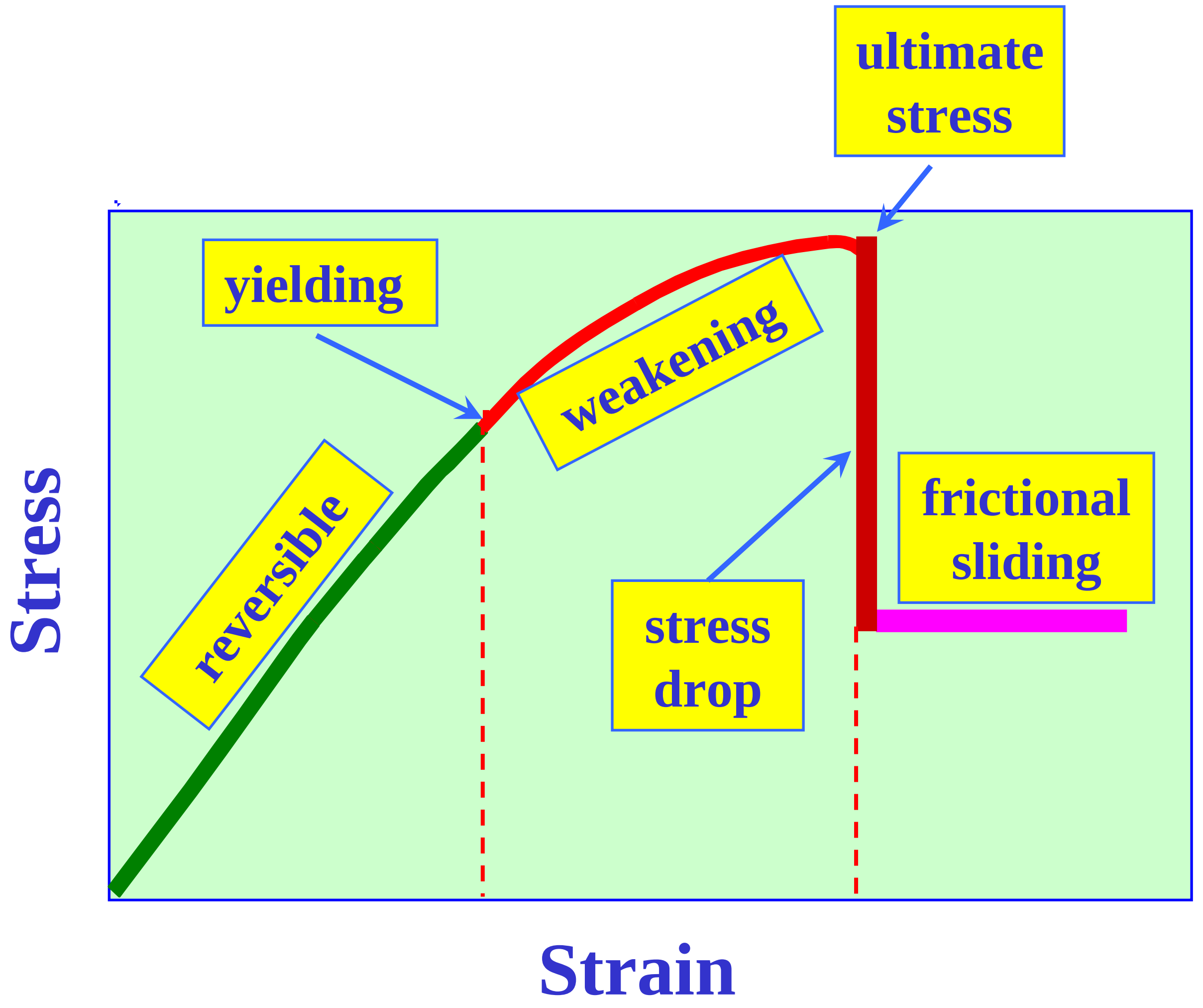
What a rheological model should do?



Physical aspects of the damage rheology model:

- **Mechanical aspect:**

Elastic moduli depend on the microcrack density through scalar damage variable (α)

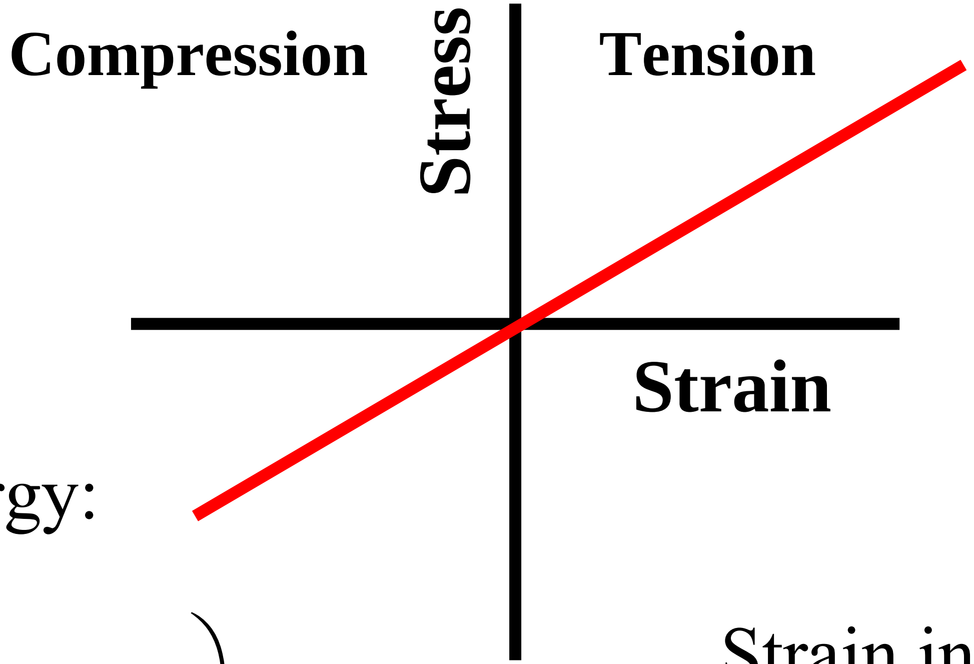
- **Kinetic aspect:**

Material damage evolves with ongoing deformation

- **Macroscopic failure:**

Convexity loss of the elastic energy potential

Linear elastic solid



Elastic energy:

$$U = \frac{1}{\rho}\left(\frac{\lambda}{2} I_1^2 + \mu I_2\right)$$

Strain invariants:

$$\mathbf{I_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{kk},}$$

$$\mathbf{I_2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = \varepsilon_{ij}\,\varepsilon_{ij}}$$

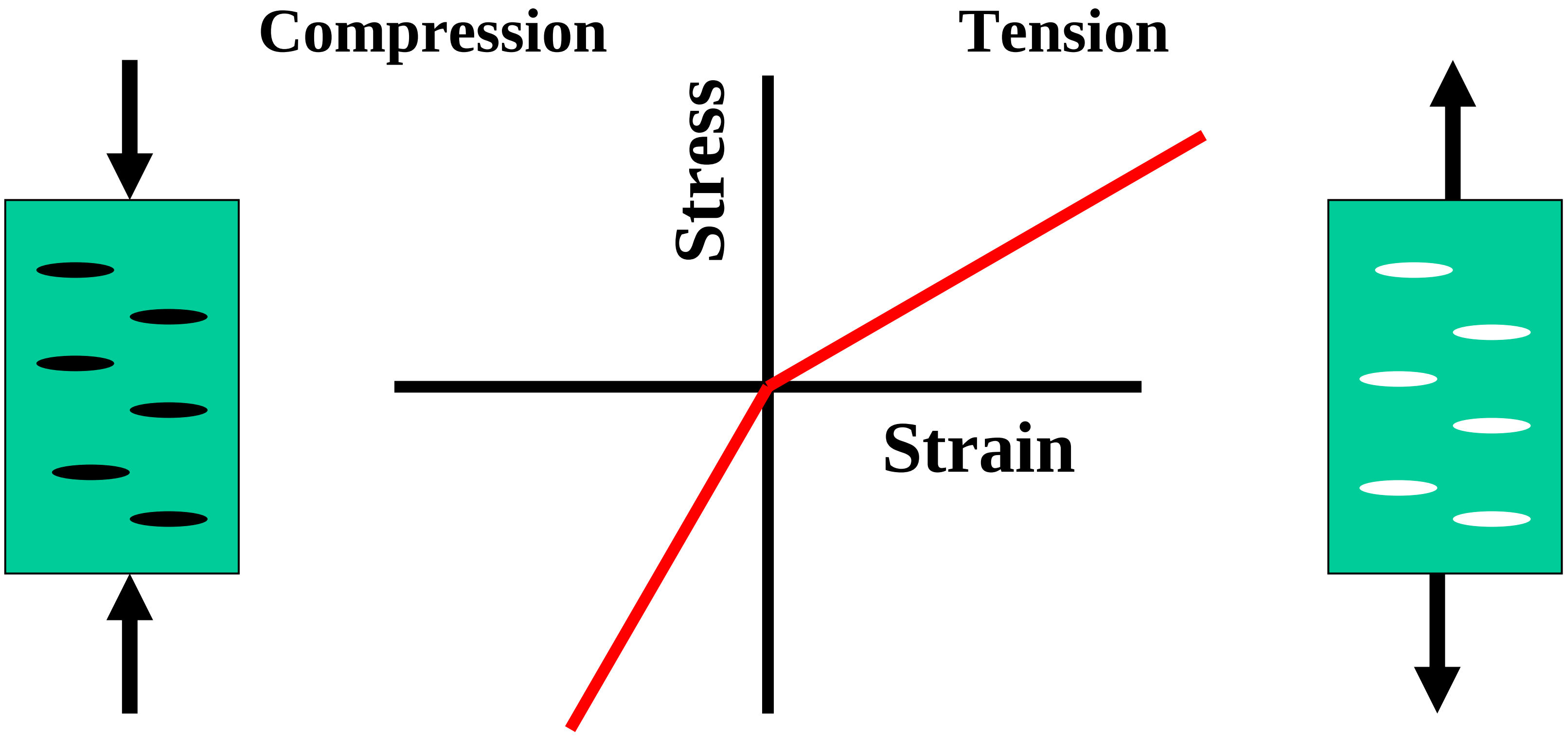
Where λ and μ are Lame constants

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

Hook law: $$\sigma_{ij} = \rho \frac{\partial U}{\partial \varepsilon_{ij}} = \lambda I_1 \delta_{ij} + 2\mu\varepsilon_{ij}$$

Mechanical aspect -
the sensitivity of the macroscopic elastic moduli to distributed cracks and to the type of loading



The macroscopic effects of distributed cracking and other types of damage require constitutive models, which exhibit non-linear stress-strain relations

The elastic energy, U, is written as:

$$U = \frac{1}{\rho}\left(\frac{\lambda}{2} I_1^2 + \mu I_2 - \gamma I_1 \sqrt{I_2}\right)$$

$$\xi = \frac{\mathbf{I_1}}{\sqrt{\mathbf{I_2}}}$$

Where λ and μ are Lame constants;
γ is an additional elastic modulus

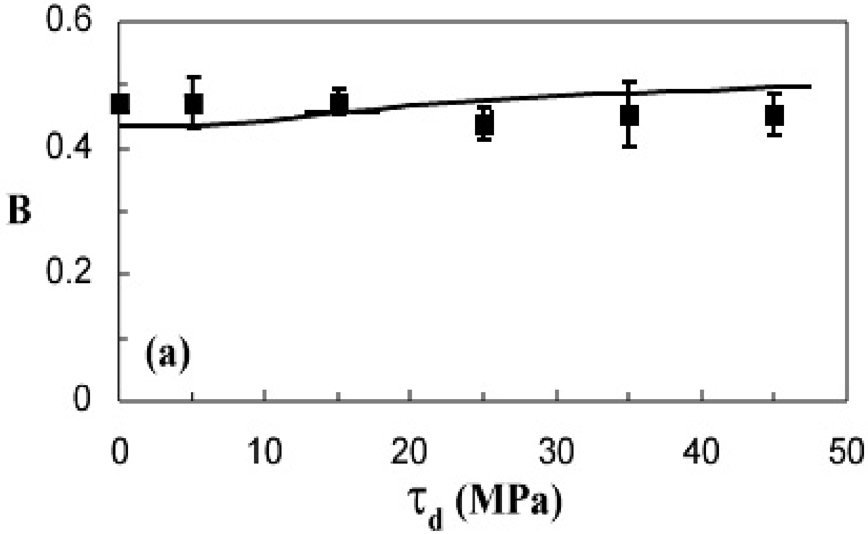
$\mathbf{I_1 = \varepsilon_{kk}}$,
$\mathbf{I_2 = \varepsilon_{ij}\,\varepsilon_{ij}}$

$$\sigma_{ij} = \rho \frac{\partial U}{\partial \varepsilon_{ij}} = \left(\lambda - \gamma \frac{\sqrt{I_2}}{I_1}\right) I_1 \delta_{ij} + \left(2\mu - \gamma \frac{I_1}{\sqrt{I_2}}\right) \varepsilon_{ij}$$

Rock dilation

Navajo sandstone

$$\ldots\sqrt{I_2}\Big) + \left(2\mu - \gamma\frac{I_1}{\sqrt{I_2}}\right)I_1$$

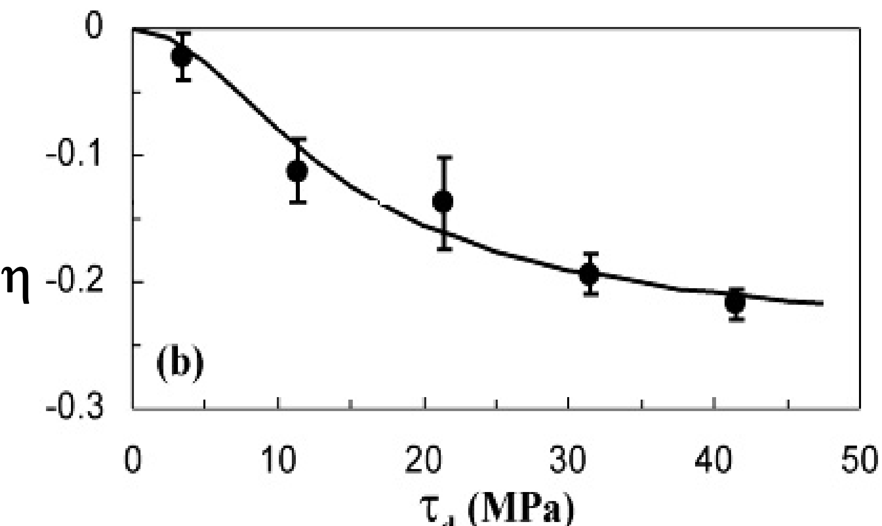


Fluid pressure change under undrained conditions

$$dP_{fluid} = B d\sigma_m + \eta d\tau_d$$

B – Skempton coefficient

$\eta \leq 0$ and $|\eta|$ increases with τ_d



Data: Lockner & Stanchits, JGR, 2002

Model: Hamiel et al., EPSL, 2005

- **Kinetic aspect:**

Material damage evolves with ongoing deformation

Elastic moduli λ, μ and γ are functions of the non-dimensional damage parameter

$\alpha(x,y,z,t)$

Thermodynamics

Free energy of a solid, F, is

$$\mathbf{F = F(T, \varepsilon_{ij}, \alpha)}$$

T – temperature, ε_{ij} – elastic strain tensor,
α – scalar damage parameter

Energy balance

$$\frac{dU}{dt} = \frac{d}{dt}\left(F + TS\right) = \frac{1}{\rho}\sigma_{ij}e_{ij} - \nabla_i J_i$$

Entropy balance

$$\frac{dS}{dt} = -\nabla_i\left(\frac{J_i}{T}\right) + \Gamma$$

Gibbs equation

$$dF = -SdT + \frac{\partial F}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial F}{\partial \alpha} d\alpha$$

The internal entropy production rate per unit mass, Γ, is:

$$\Gamma = -\frac{J_i}{\rho T^2}\nabla_i T + \frac{1}{T}\sigma_{ij}e_{ij} - \frac{1}{T}\frac{\partial F}{\partial \alpha}\frac{d\alpha}{dt} \geq 0$$

To provide the positive entropy production the equation of damage evolution is expressed as:

$$\frac{d\alpha}{dt} = -C\frac{\partial F}{\partial \alpha}$$

Assuming linear relation:

$$\lambda = \lambda_0 = \text{const}; \quad \mu = \mu_0 + \alpha\xi_0\gamma_r; \quad \gamma = \alpha\gamma_r$$

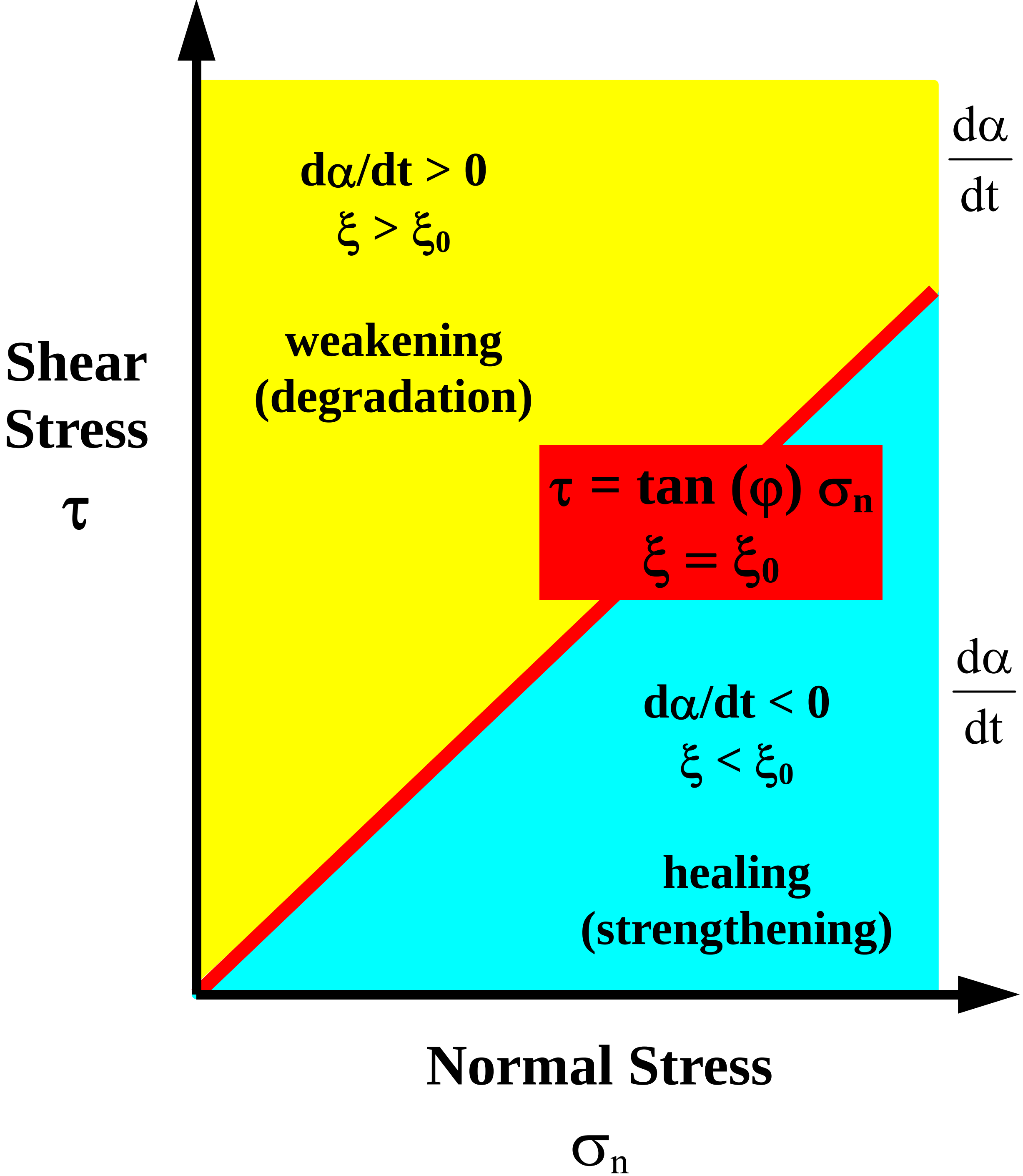
The final form of the equation for damage evolution is:

$$\frac{d\alpha}{dt} = \begin{cases} C_d \cdot I_2\left(\xi - \xi_0\right) & \text{for} \quad \text{degradation} \\ C_1 \cdot \exp\left(\frac{\alpha}{C_2}\right) \cdot I_2\left(\xi - \xi_0\right) & \text{for} \quad \text{healing}, \end{cases}$$

The damage rheology has two types of functional coefficients:

(1) a “generalized internal friction” (ξ_0) separates between states associated with material weakening (degradation) and healing (strengthening)

(2) damage rate coefficients for positive (degradation) and negative (healing)



$$\frac{d\alpha}{dt} = C_d \cdot I_2 \left(\xi - \xi_0\right)$$

$$\xi_0 = \frac{-\sqrt{3}}{\sqrt{2q^2\left(\frac{\lambda_0}{\mu_0} + \frac{2}{3}\right)^2 + 1}}$$

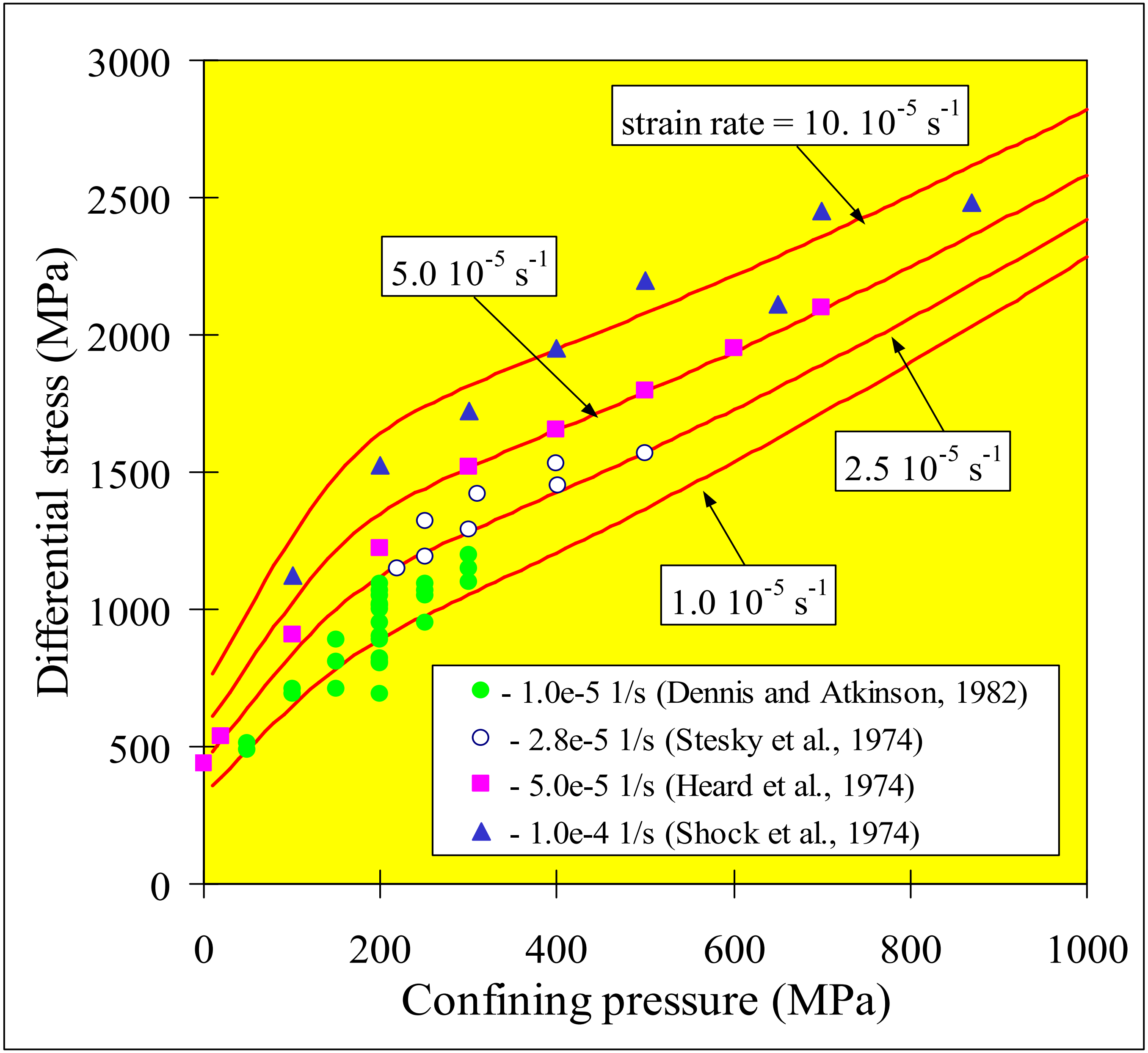
where $q = \frac{\sin(\varphi)}{1 - \sin(\varphi)/3}$.

$$\frac{d\alpha}{dt} = C_1 \cdot \exp\left(\frac{\alpha}{C_2}\right) \cdot I_2 \left(\xi - \xi_0\right)$$

$$\boxed{\xi = \frac{I_1}{\sqrt{I_2}}}$$

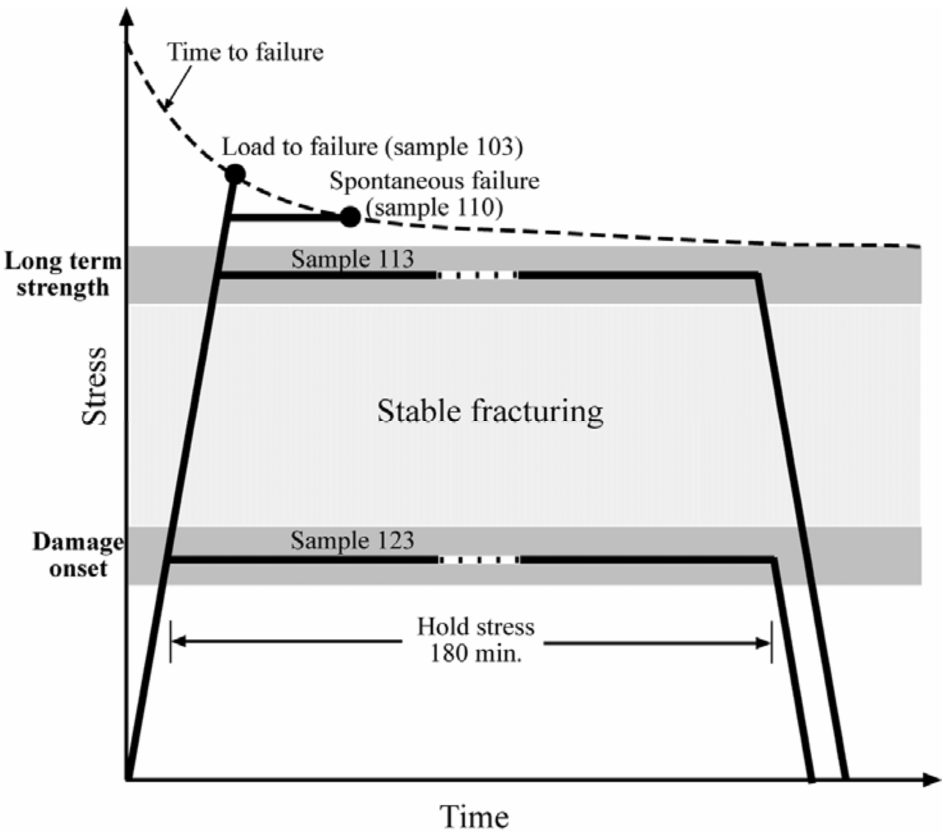
Strain invariant ratio

Ultimate stress for Westerly Granite

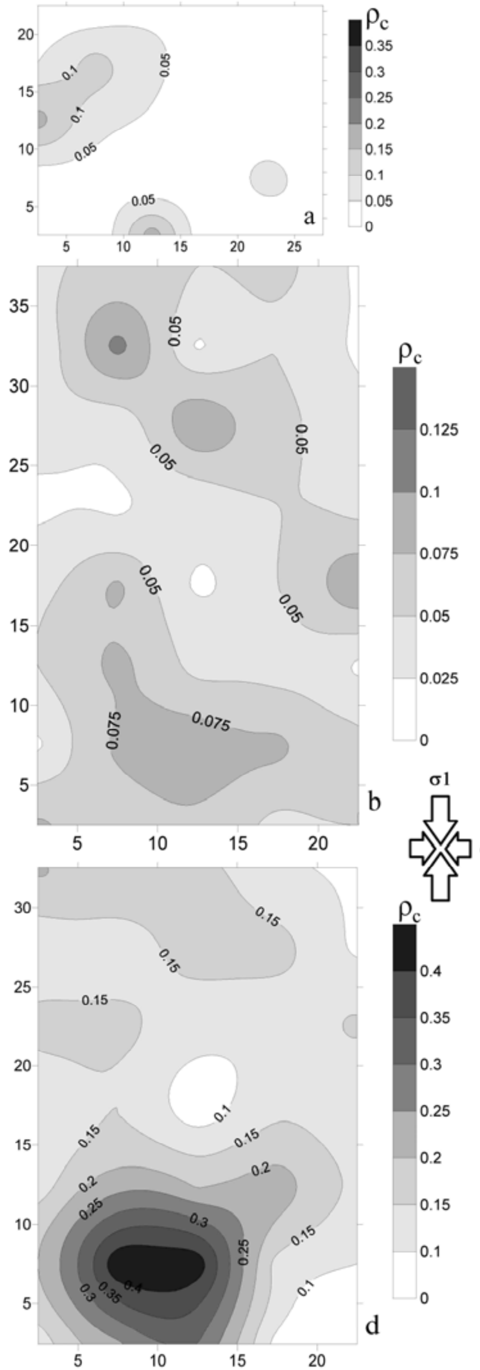


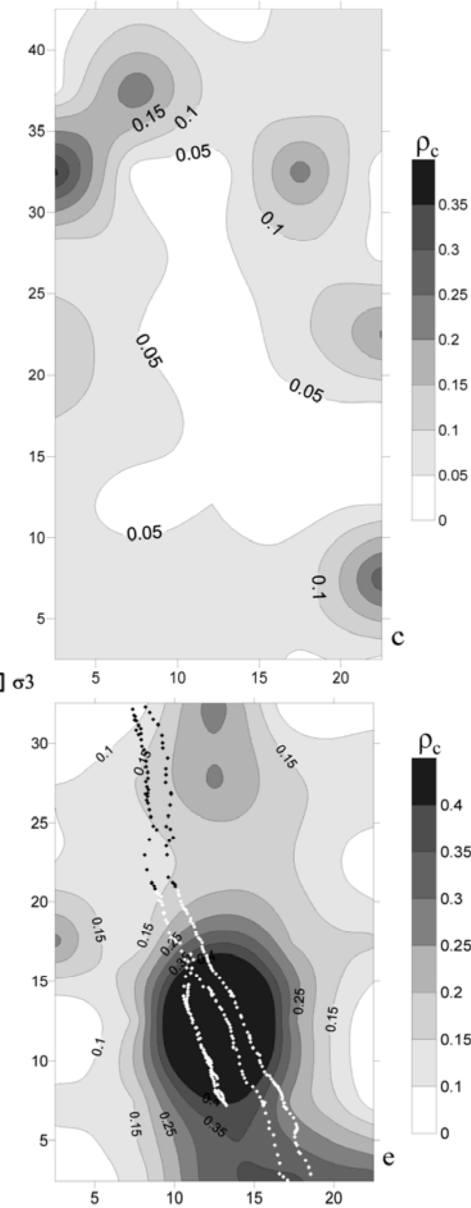
Stable and unstable damage evolution

Mount Scott Granite



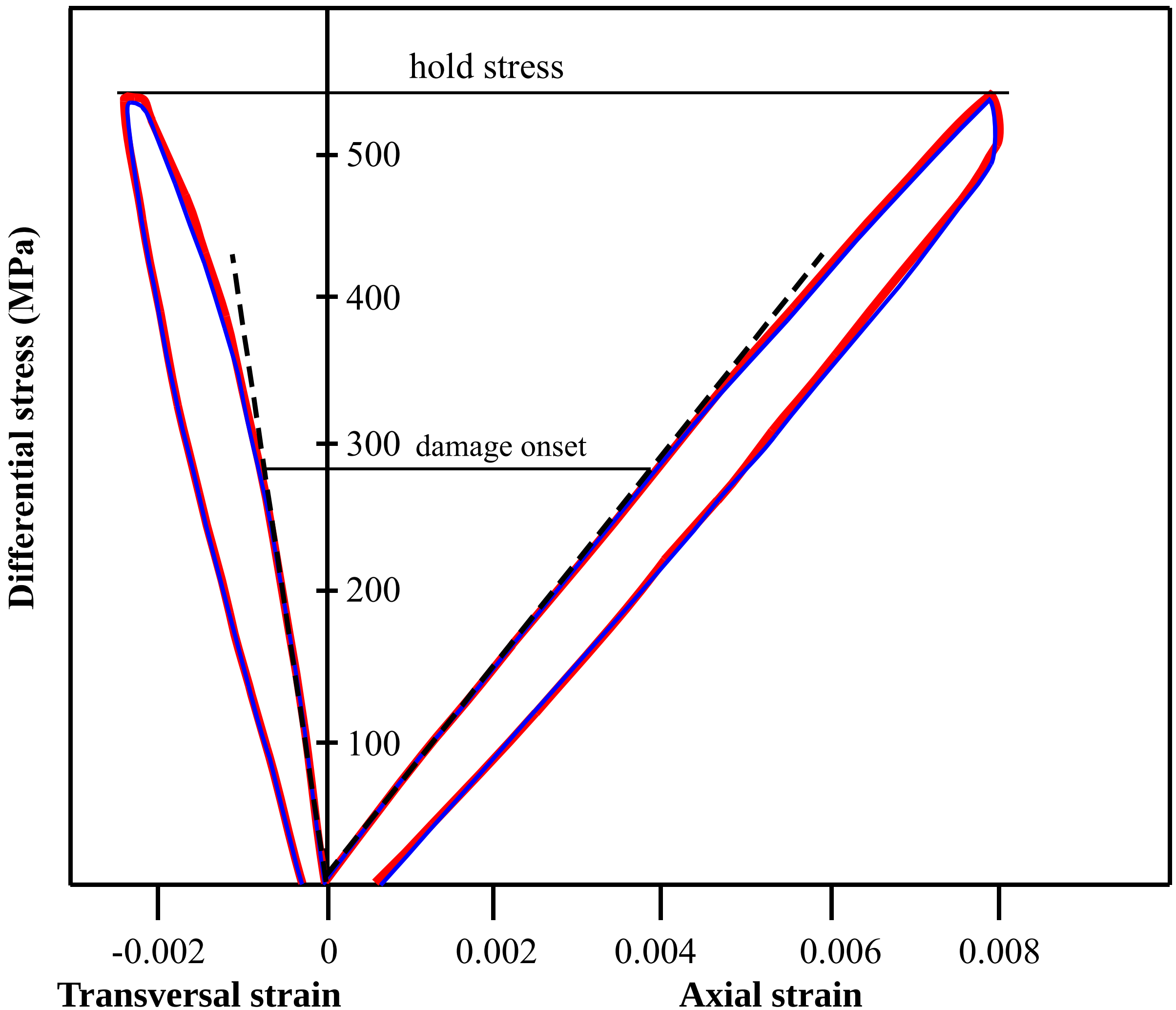
Measured crack density

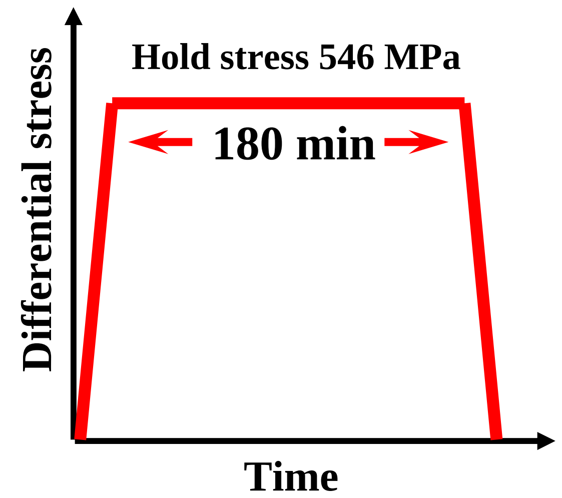




Data from *Katz and Reches* (2004)

Mount Scott Granite





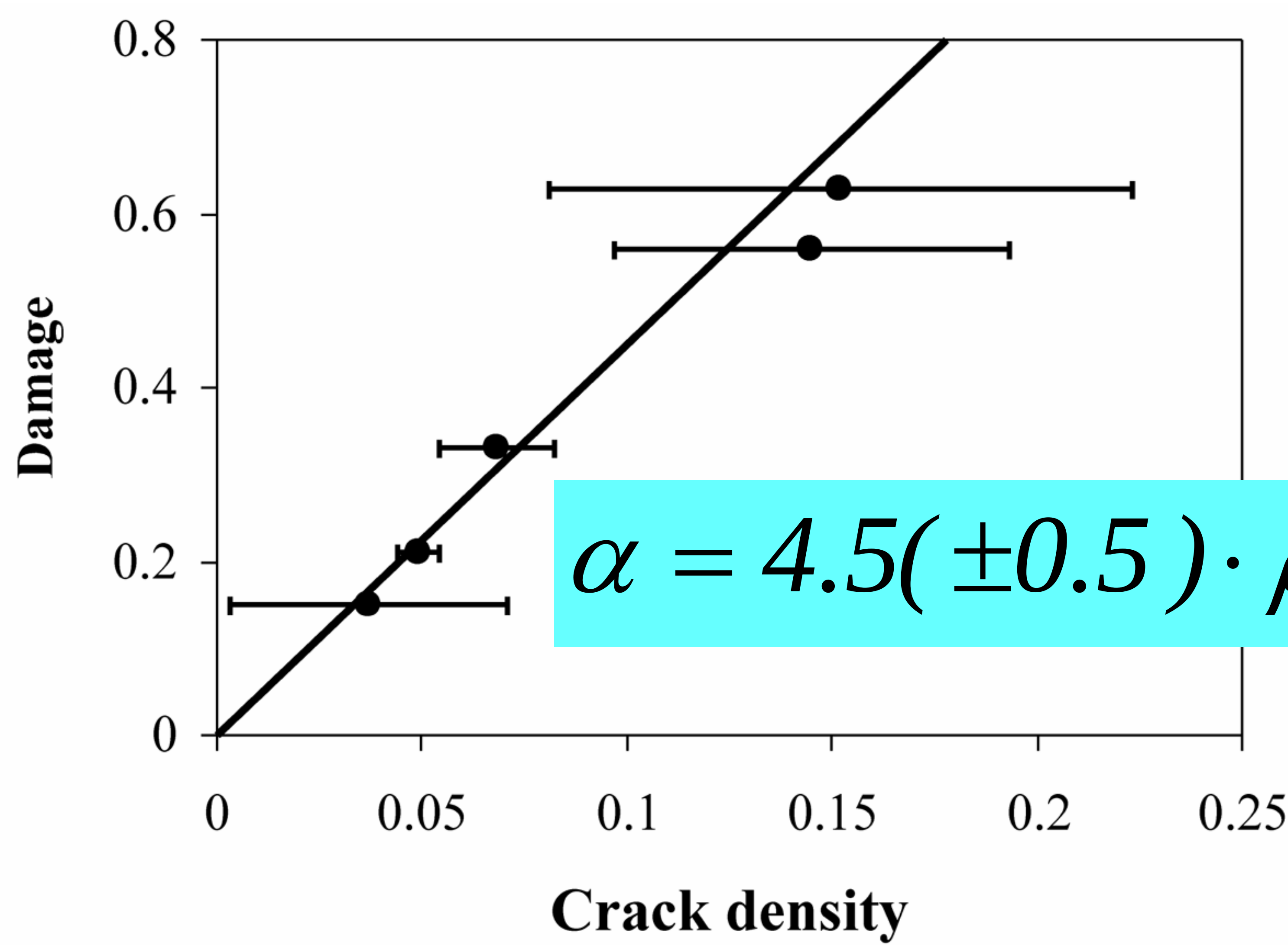
μ_0 = 3.0 10^{10} Pa,
C_v = 1.5 10^{-11} Pa^{-1},

R = 0.45

Data from *Katz and Reches* (2004), sample #109

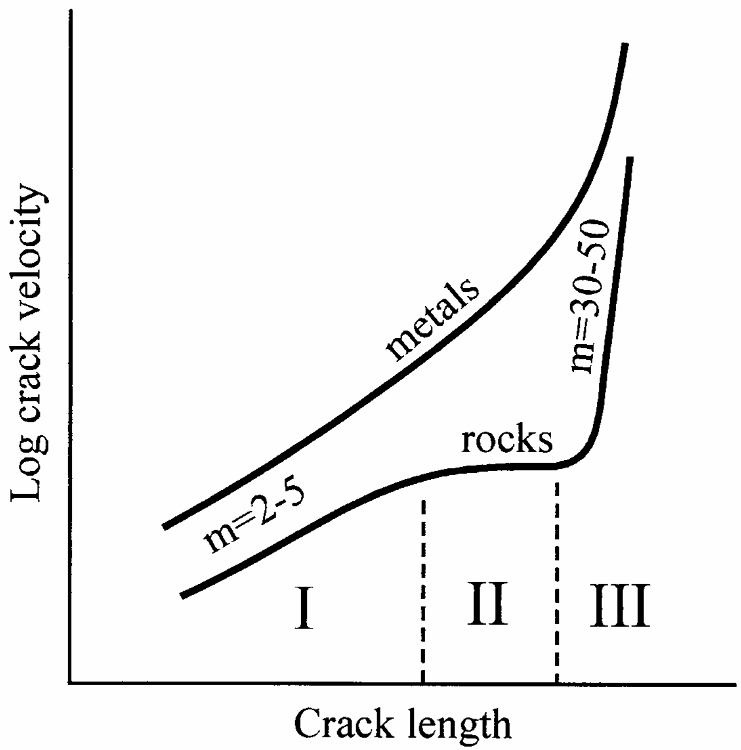
Model from Hamiel *et al*., (2006)

microcrack density vs. damage variable



$$\alpha = 4.5(\pm 0.5)\cdot \rho_c$$

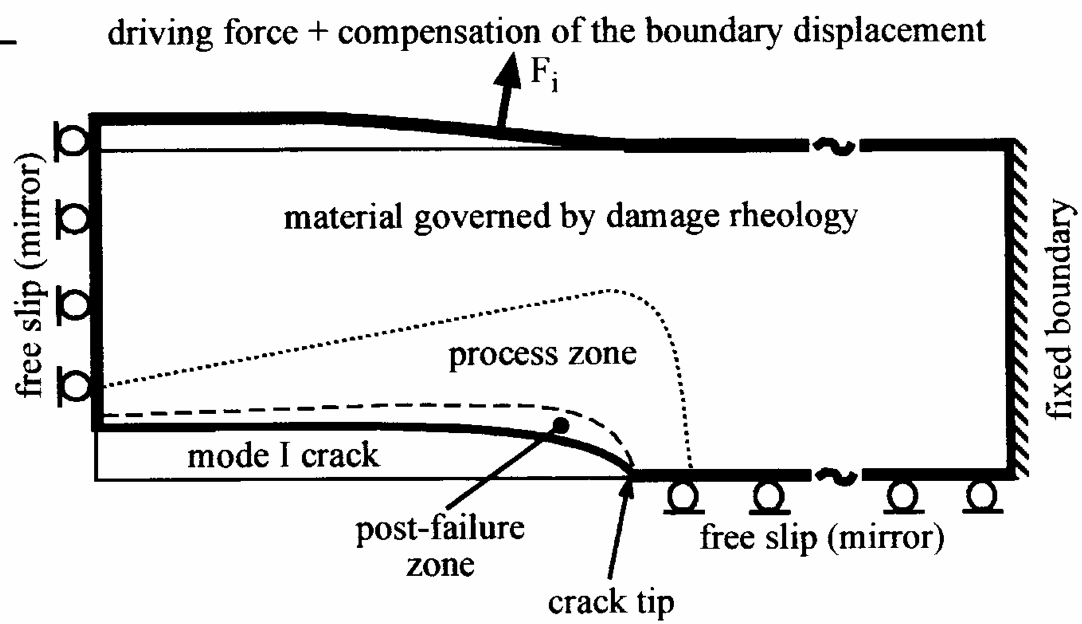
Scaling of fracture length and distributed damage

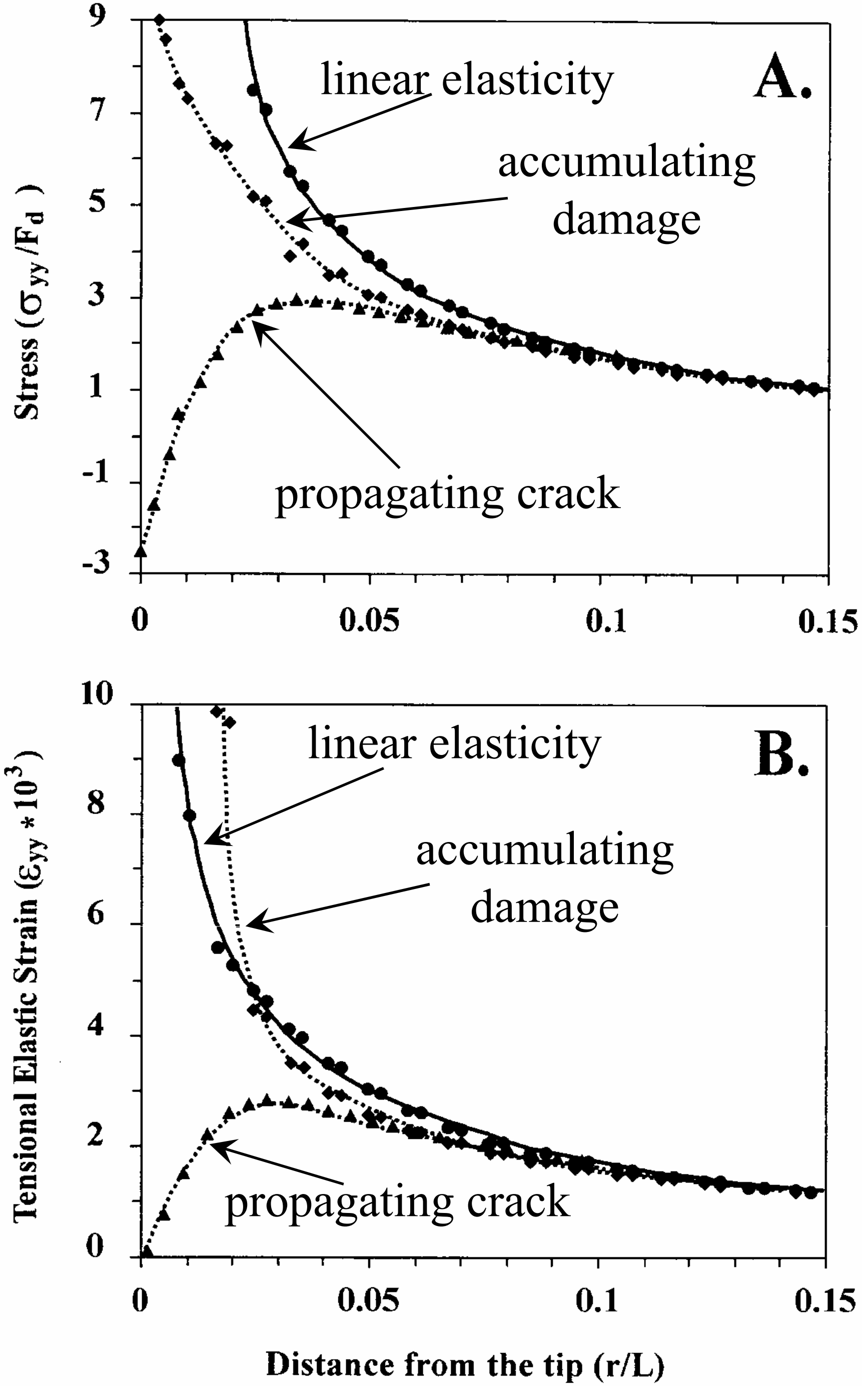


Typical relation between crack length and crack growth rate

Charles, 1958
Paris & Erdogan, 1963

The problem set-up for numerical simulation of crack propagation.



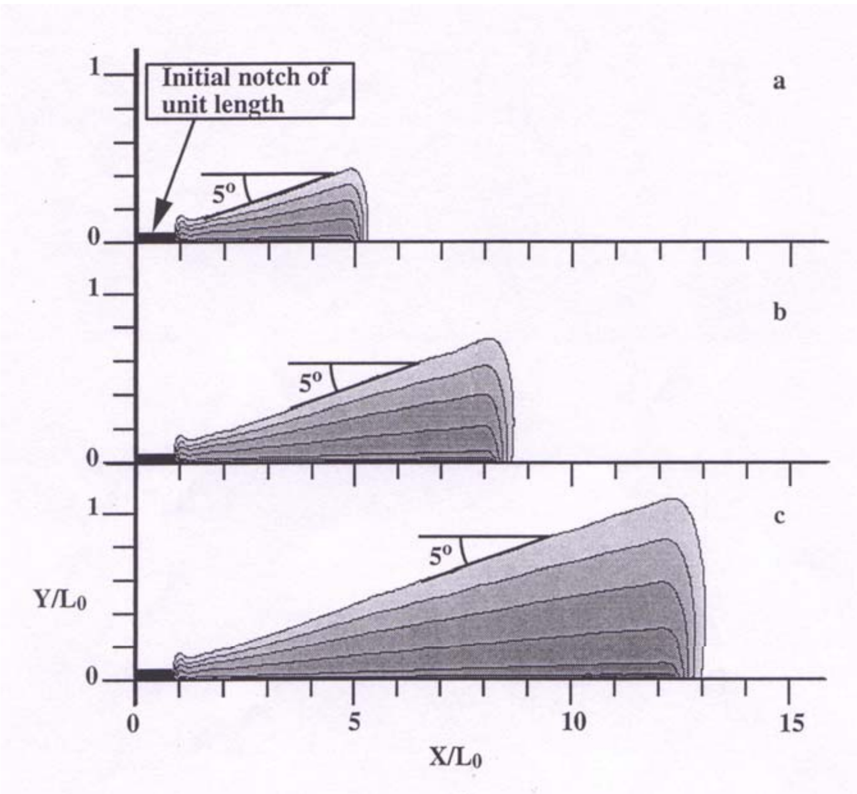


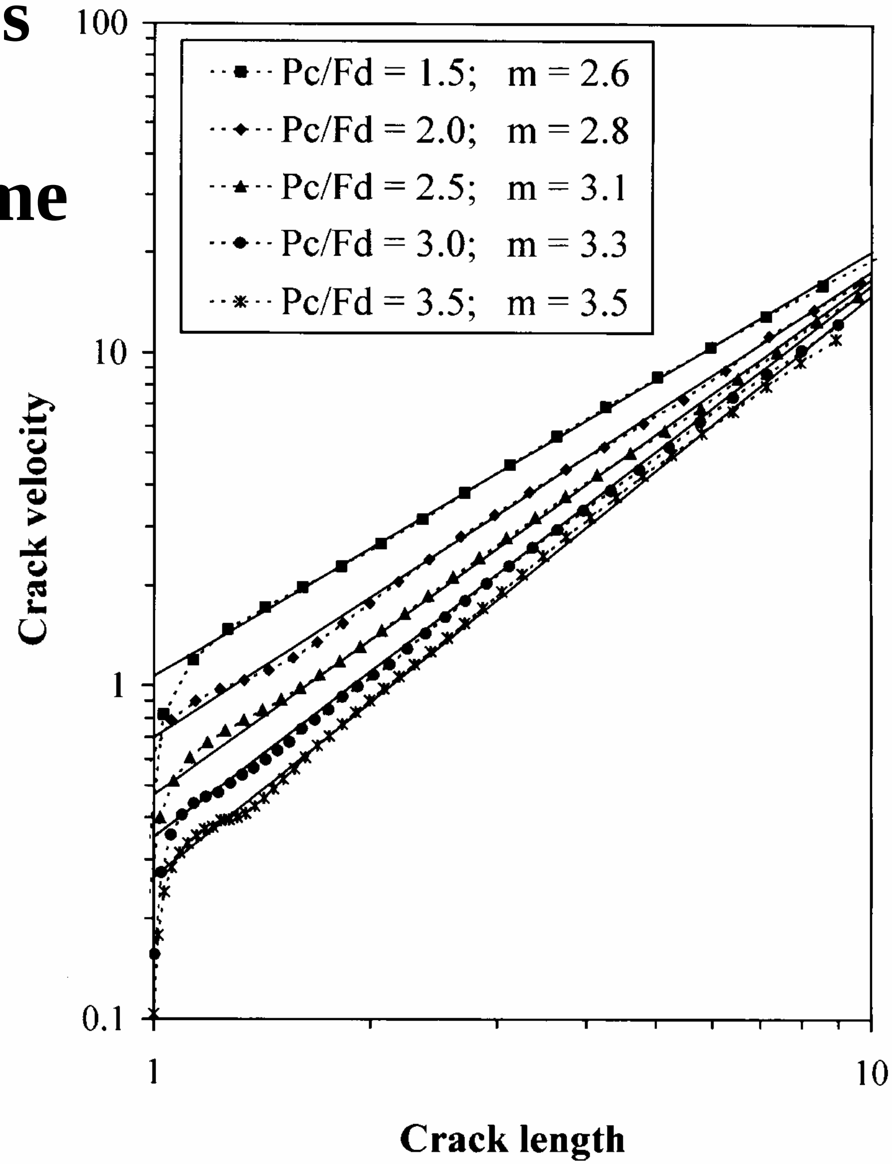
The process zone around the crack eliminates tip singularity for stress and strain distribution (triangles).

Circles represent the stress and strain distributions prior to the onset of damage; this configuration fits perfectly an analytical solution (heavy line).

Diamonds correspond to the stress and strain distributions around a crack surrounded by a process zone prior to the onset of crack propagation.

Self-similar shape of a process zone around propagating crack in the quasi-static regime

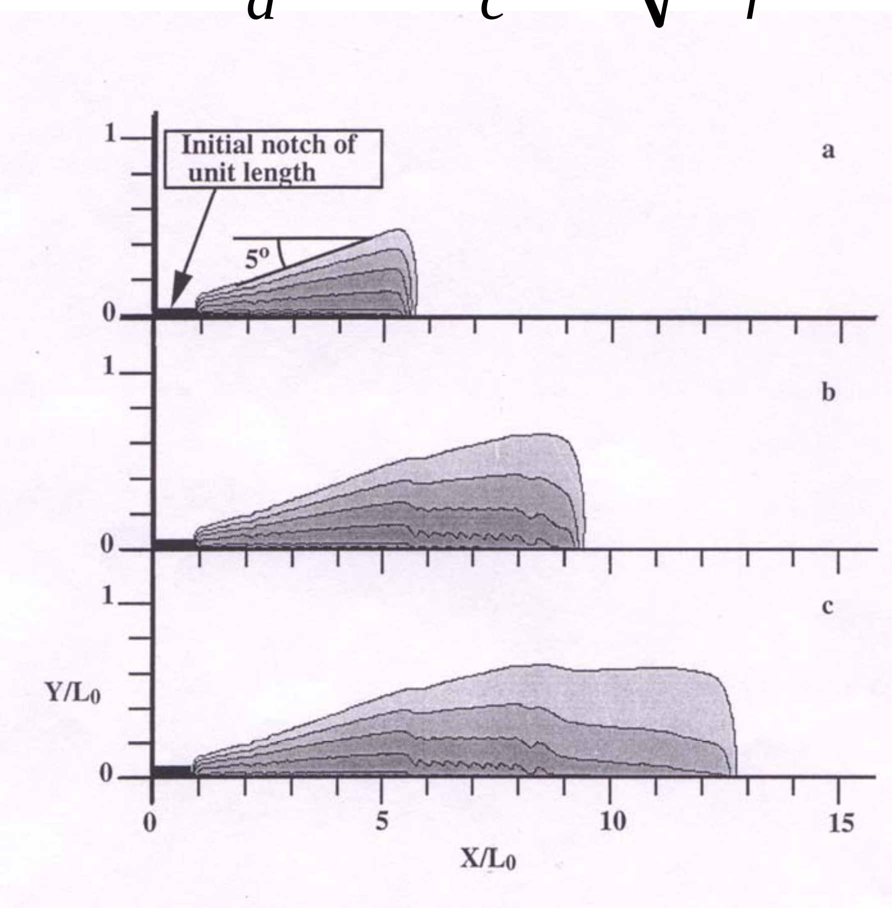


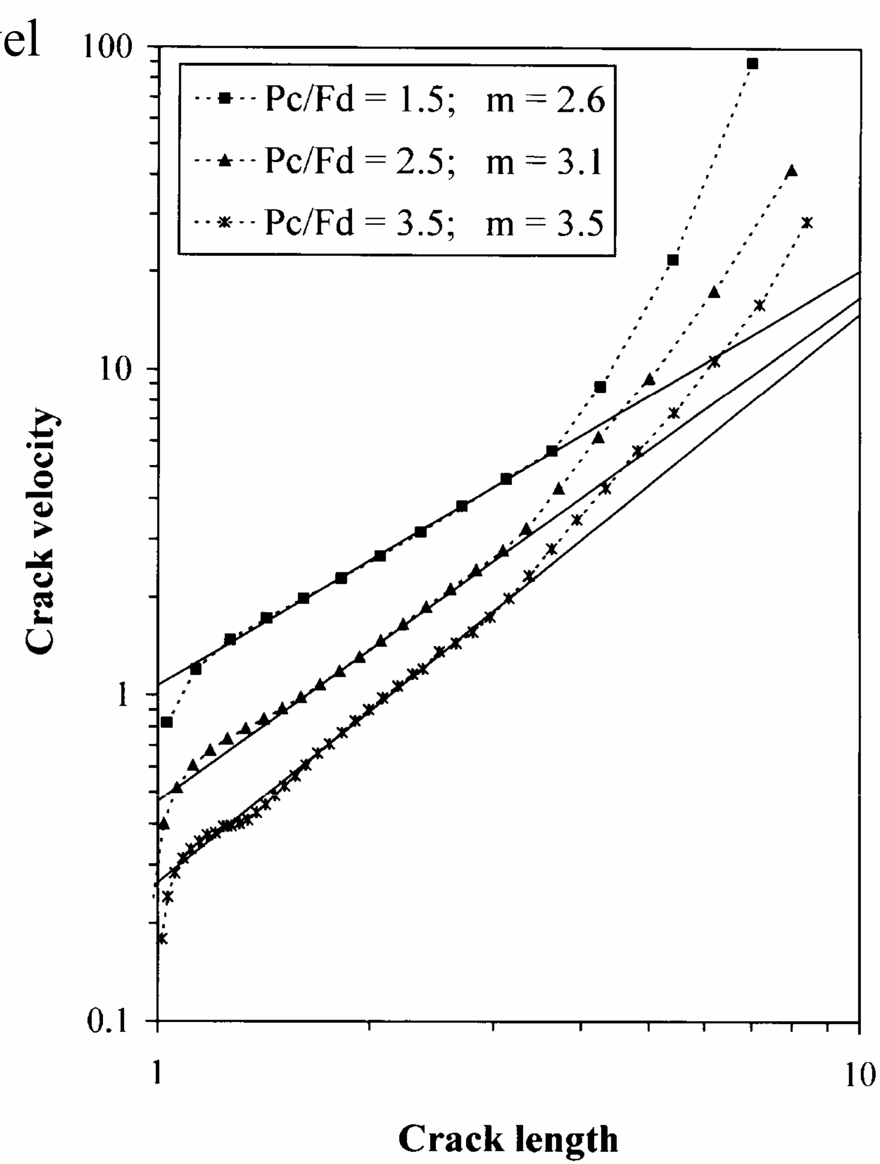


Dynamic weakening

The critical level of damage for propagation of the instability is lower than the initiation level by a dynamic weakening factor:

$$\alpha_d = \alpha_c - \sqrt{\tau_r \dot{\alpha}}$$

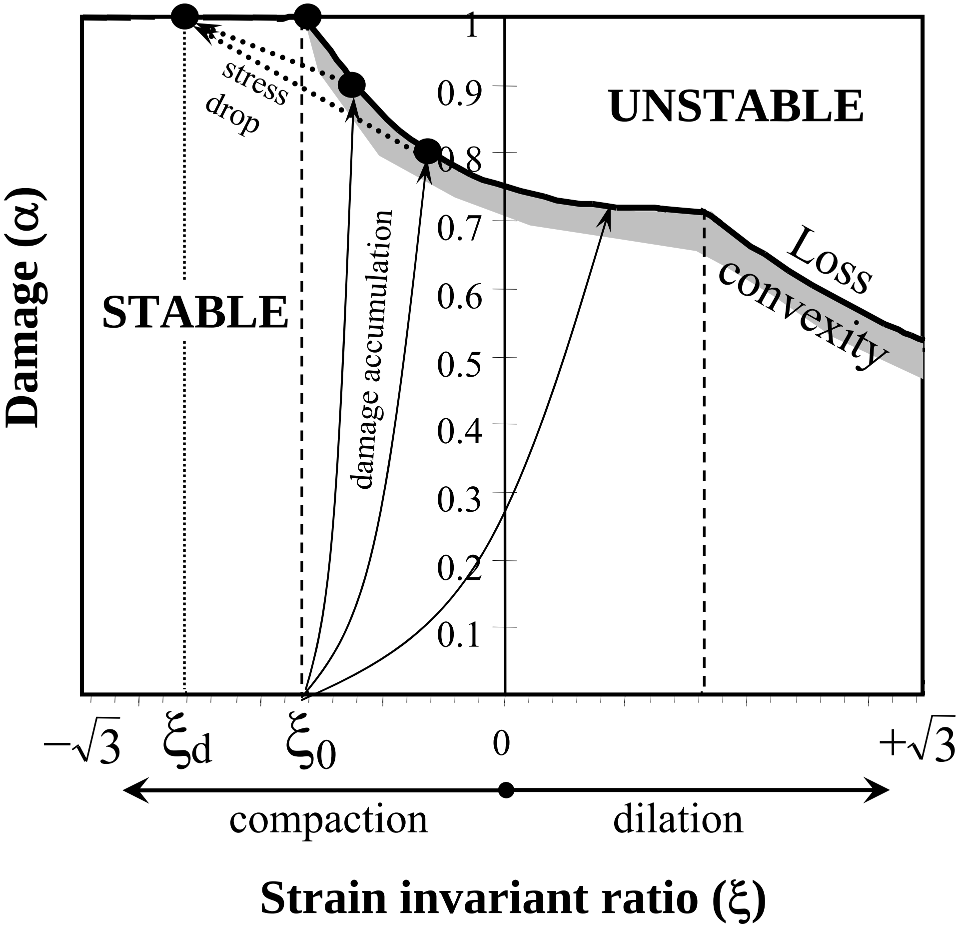




Macroscopic failure:

Convexity loss of the elastic energy potential

$$U = \frac{1}{\rho}\left(\frac{\lambda}{2} I_1^2 + \mu I_2 - \gamma I_1 \sqrt{I_2}\right)$$

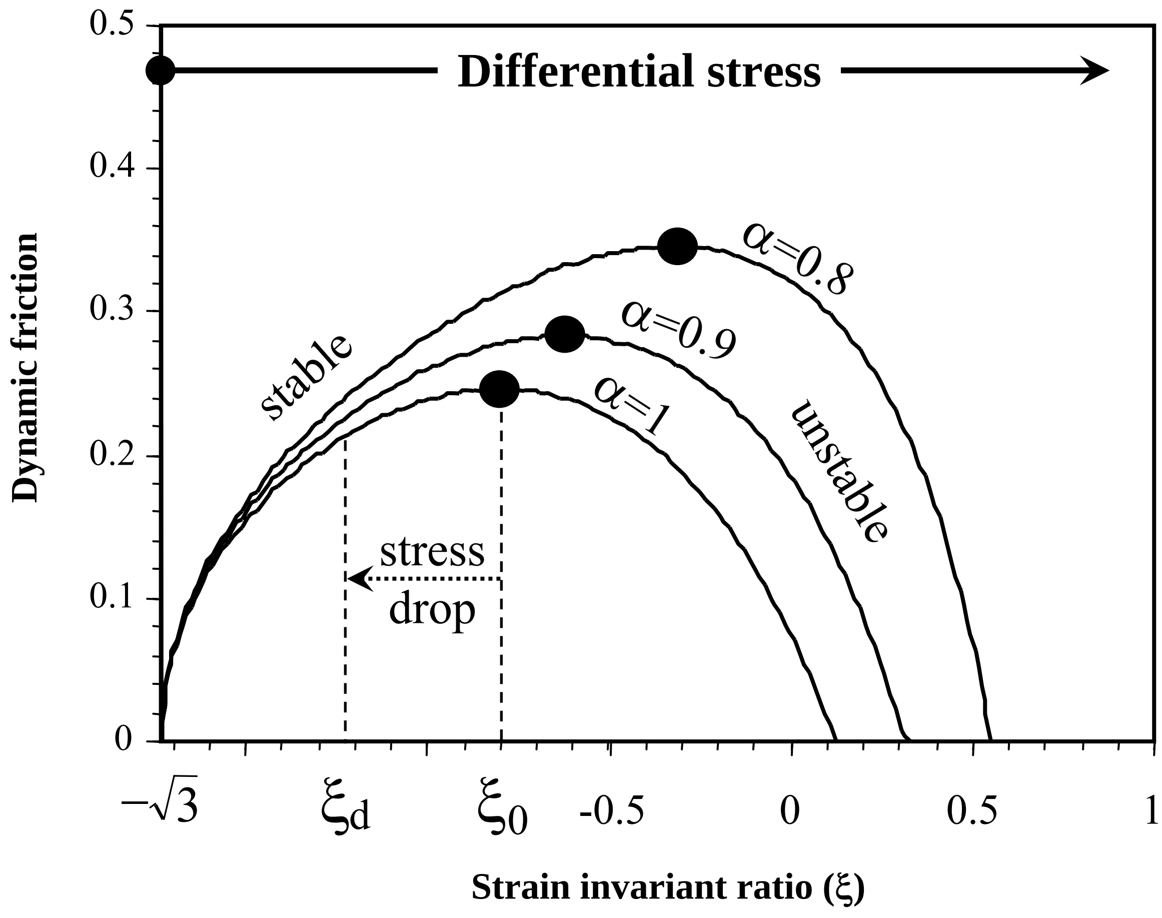


$$\lambda = const.$$

$$\mu = \mu_0 + \alpha \xi_0 \gamma_r \downarrow$$

$$\gamma = \alpha \gamma_r \uparrow$$

Effective residual or dynamic coefficient of friction after failure versus the strain invariant ratio for given values of the damage parameter α=0.8, 09, 1.



Local stress drop

Similarly to models assuming constant dynamic friction during simulated seismic events (e.g., Ben-Zion & Rice, 1993; Ben-Zion, 1996; Zöller et al., 2006), we define a constant ξ_d value using a weight factor, *w*, as a model parameter:

$$\xi_d = (1 - w) \cdot \xi_0 - w \cdot \sqrt{3} = Const$$

This condition is equivalent to the yielding condition relating the stress invariants J_1 and J_2 for a given α and ξ values:

$$Y(\sigma_{ij}) = J_1 - f(\alpha, \xi) \cdot \sqrt{J_2} < 0$$

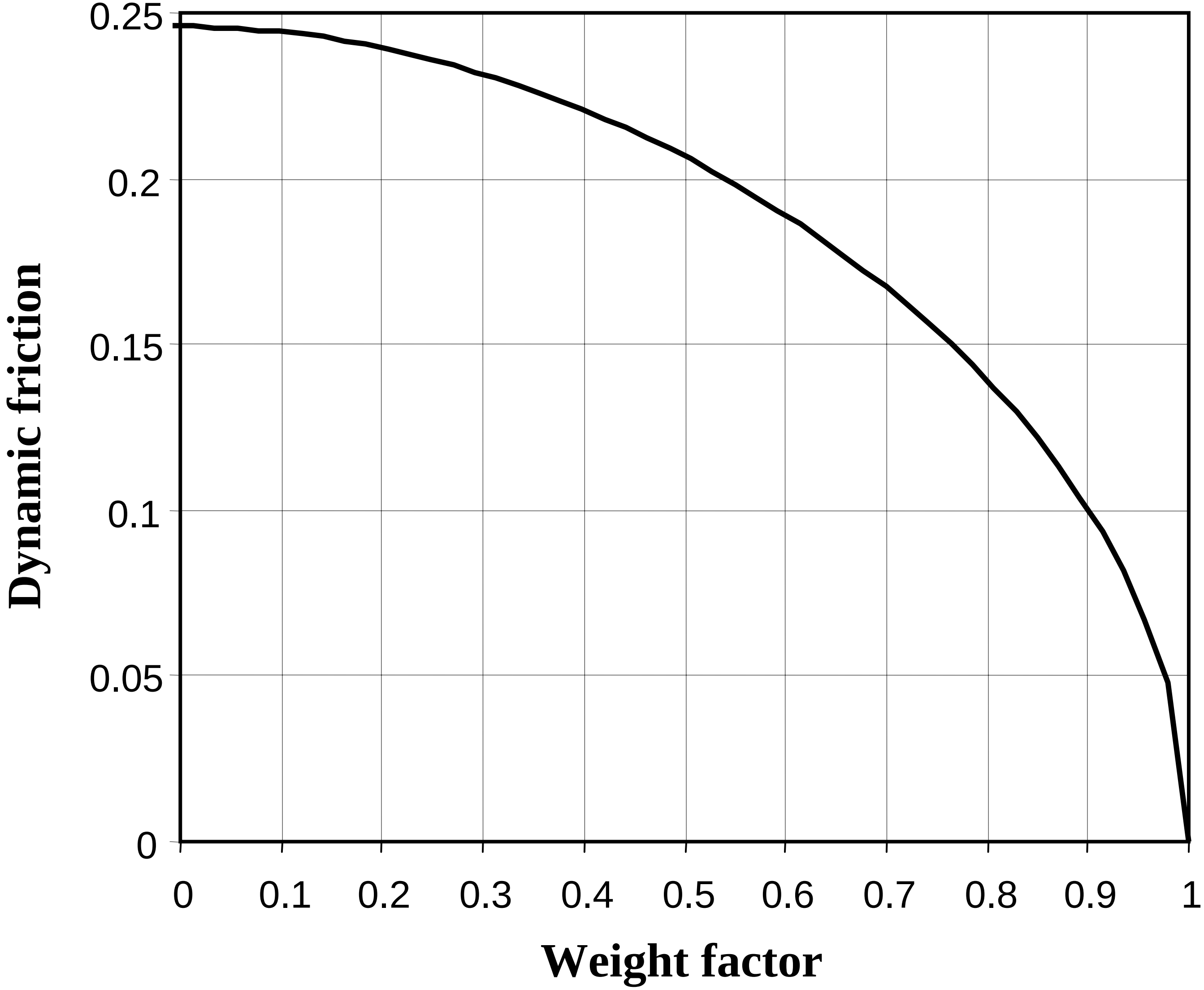
For positive values of the yielding function ($Y(\sigma_{ij}) \geq 0$) the system is unstable and plastic strain is accumulated. This mathematical formulation of the problem is equivalent to the classical Drucker-Prager model (Drucker & Prager, 1952), which generalizes the classical Coulomb yield condition for a cohesionless material (e.g., Collins & Houlsby, 1997; Hill, 1998).

Connection between the stress invariant ratio and the strain invariant ratio and damage:

$$f(\alpha,\xi)=\frac{J_1}{\sqrt{J_2}}=\frac{(3\lambda+2\mu)\xi-3\gamma-\gamma\xi^2}{(2\mu-\gamma\xi)\sqrt{1-\xi^2/3}}$$

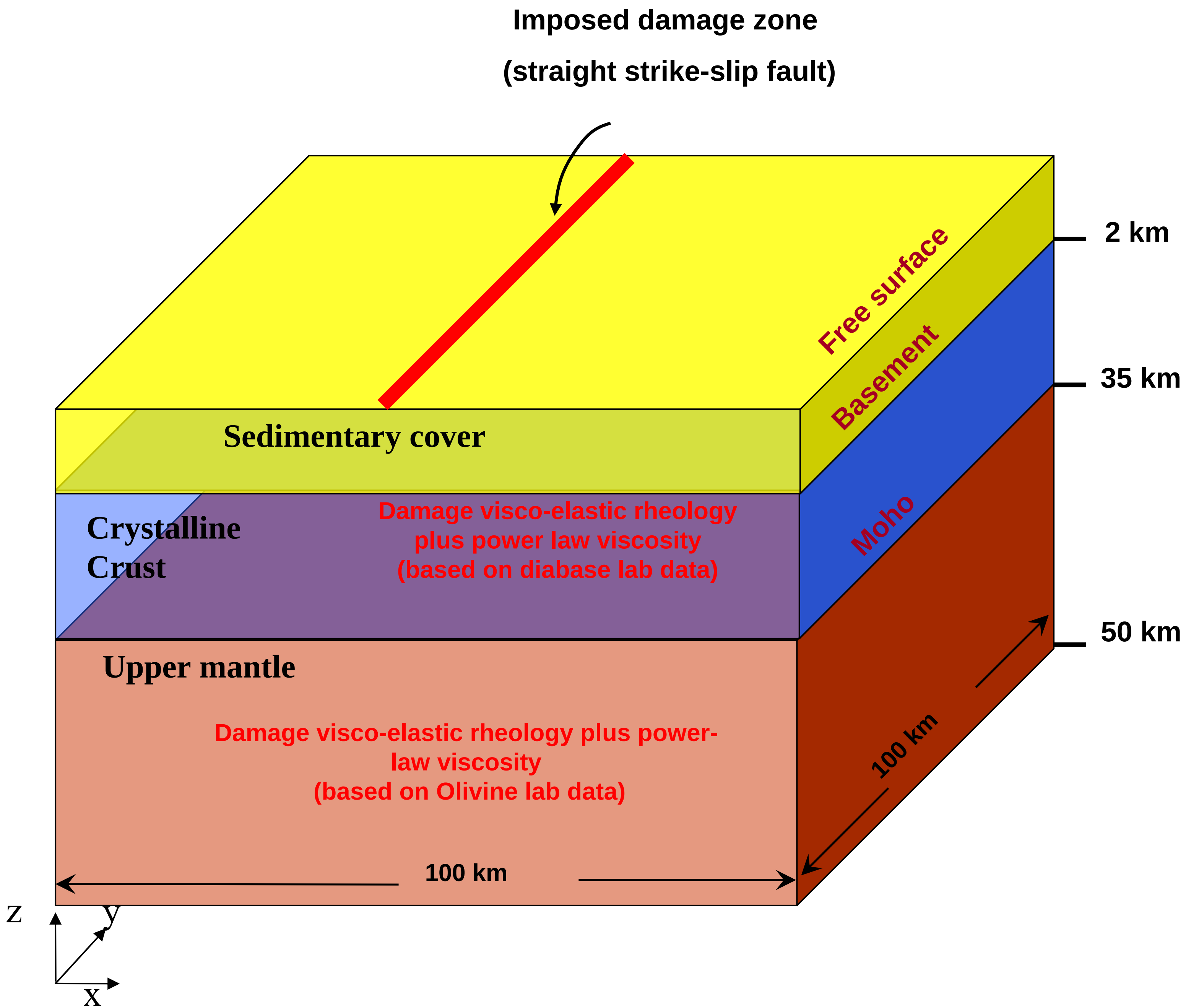
The equation describing the local stress drop in a failed element, where the conditions for convexity loss is met and the damage rheology parameters ξ and α change their values, from those corresponding to the onset of the instability to $\xi=\xi_d$ and $\alpha=1$, is:

$$d\sigma_{ij}=d\sigma_{ij}^{e}-A\frac{d\sigma_{kl}^{e}\cdot\left[\left(1+\frac{1}{3}f^{2}\right)\cdot\delta_{kl}-df\cdot\frac{\sigma_{kl}}{\sqrt{J_2}}\right]}{3+f^{2}}\cdot\left[\left(1+\frac{1}{3}f^{2}\right)\cdot\delta_{ij}-f\cdot\frac{\sigma_{ij}}{\sqrt{J_2}}\right]$$

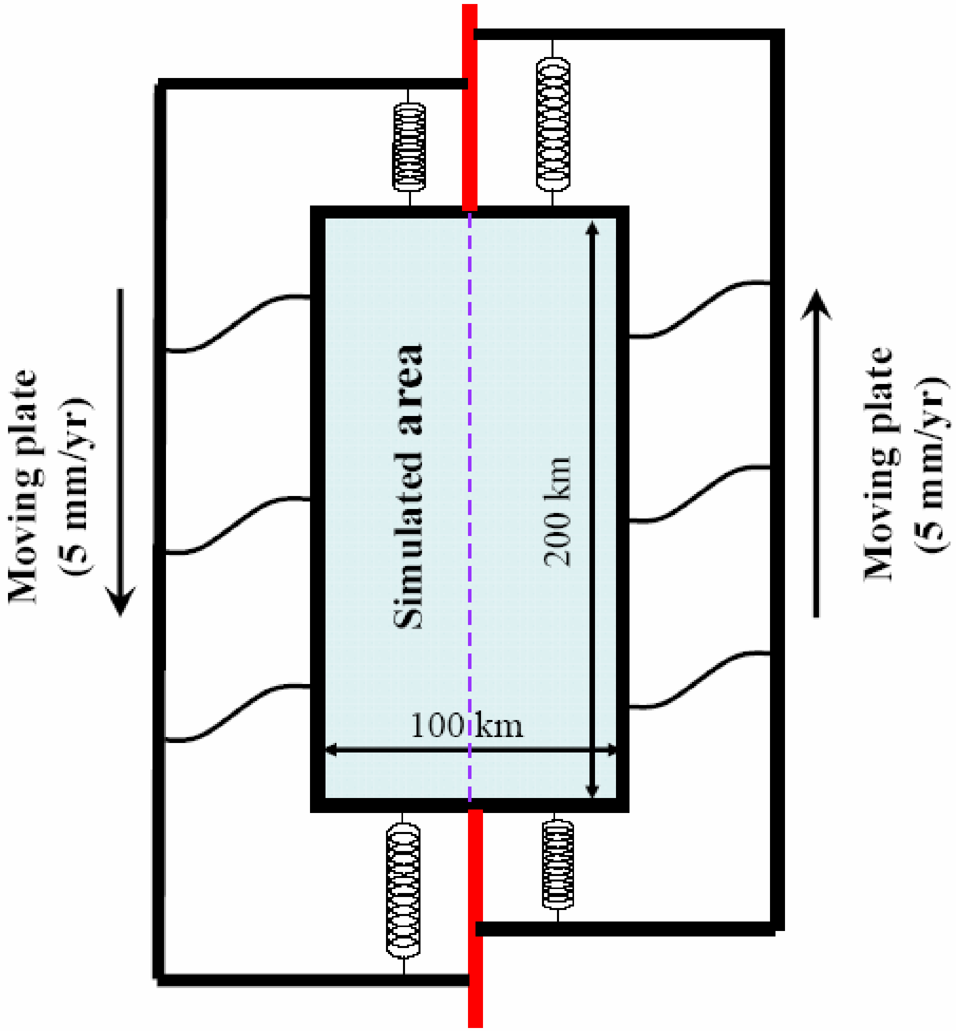


0.25
0.2
0.15
0.1
0.05
0
Dynamic friction
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Weight factor

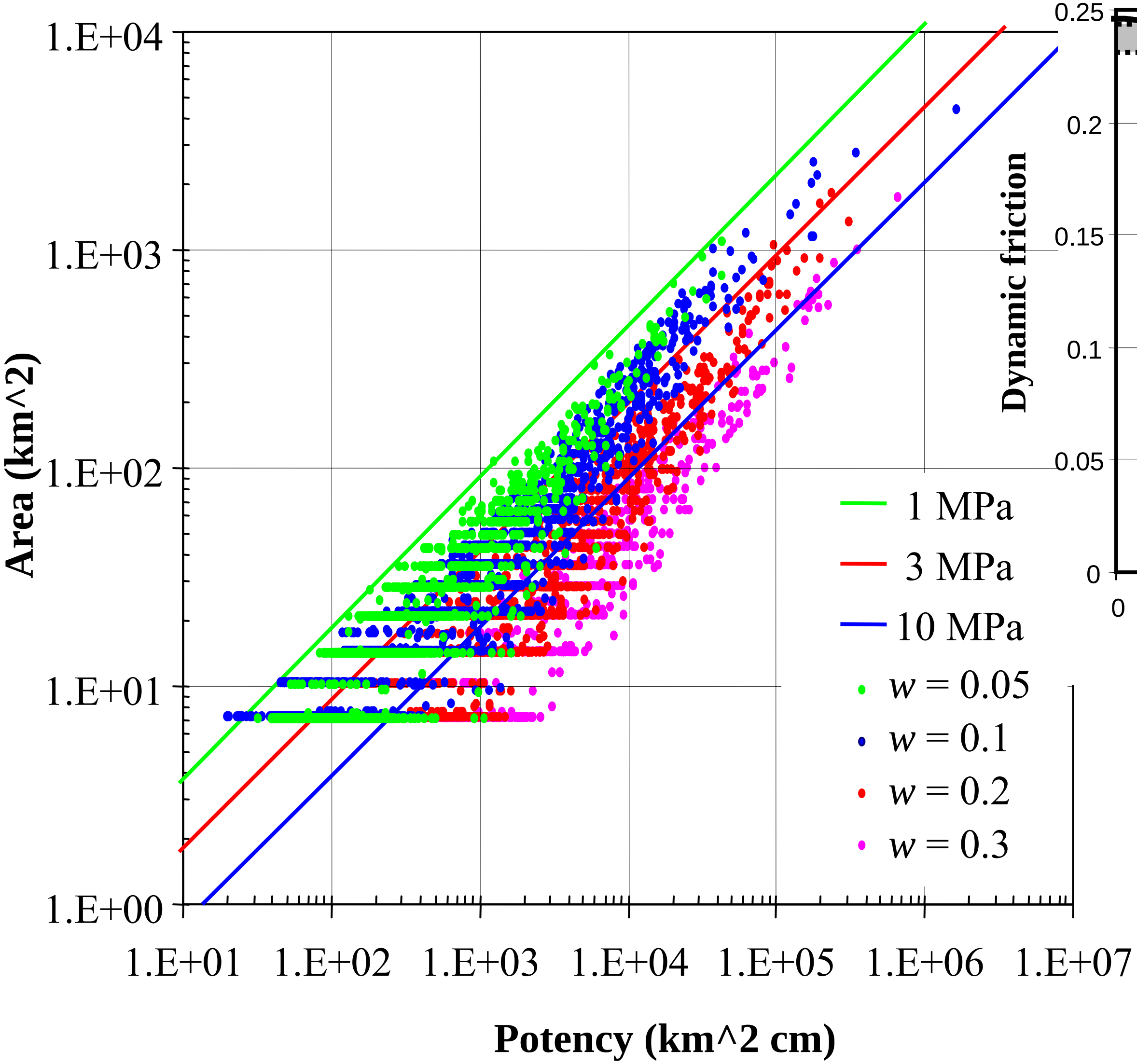
3-D lithospheric structure used in the numerical simulations

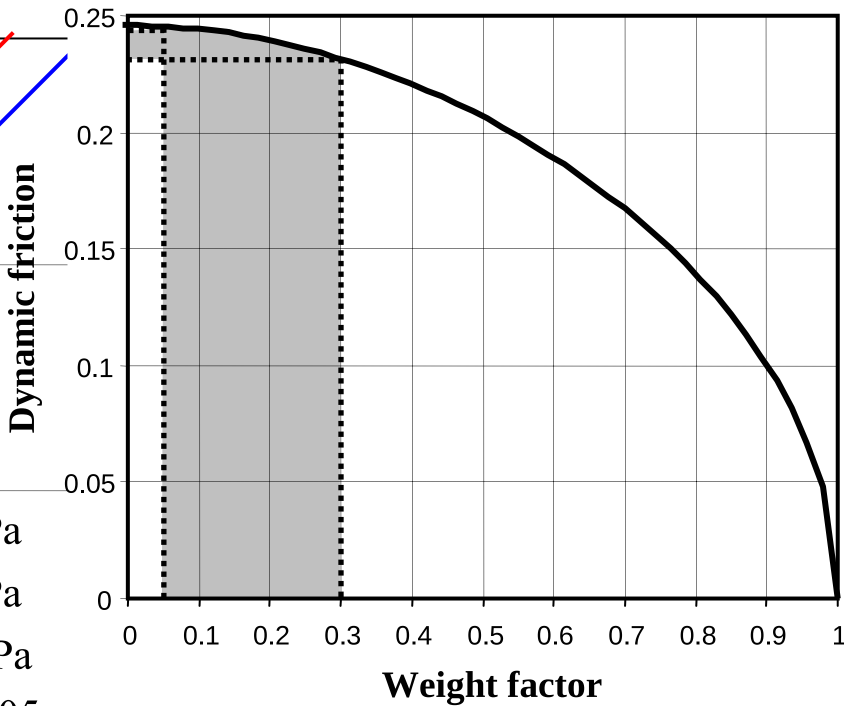


A schematic diagram illustrating the generalized boundary conditions corresponding to a constant plate motion far from the simulated model region.



Shift of the potency-area scaling relation for different values of the weight factor (*w*)

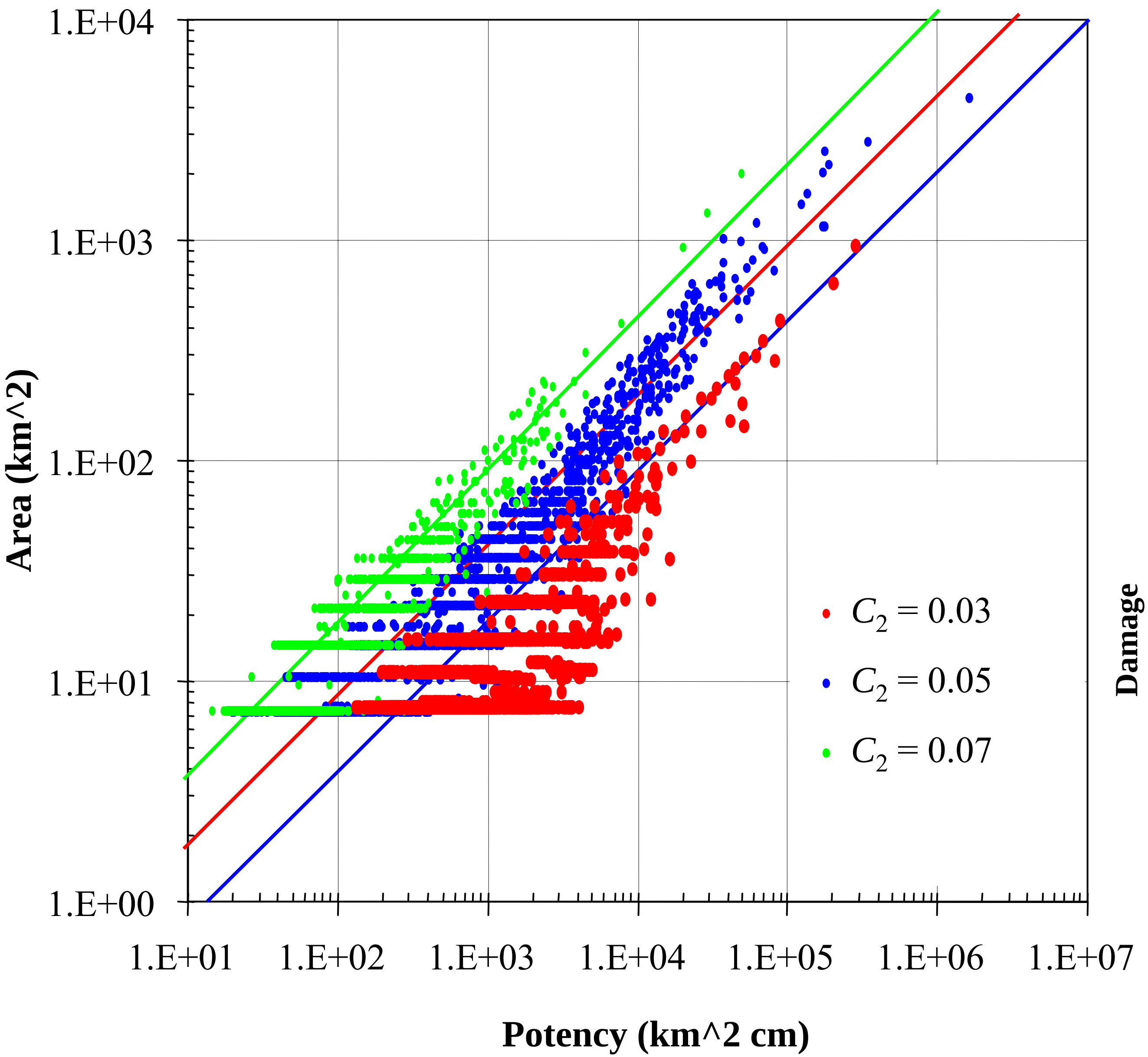




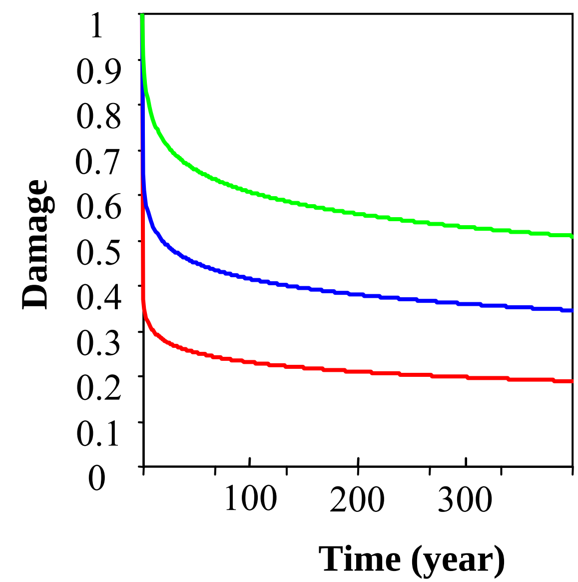
μ - shear modulus

$\Delta\sigma$ - stress drop

Potency-area scaling relation for the model with different healing rate parameter C_2 ($C_1 = 10^{-10}$ s^{-1})

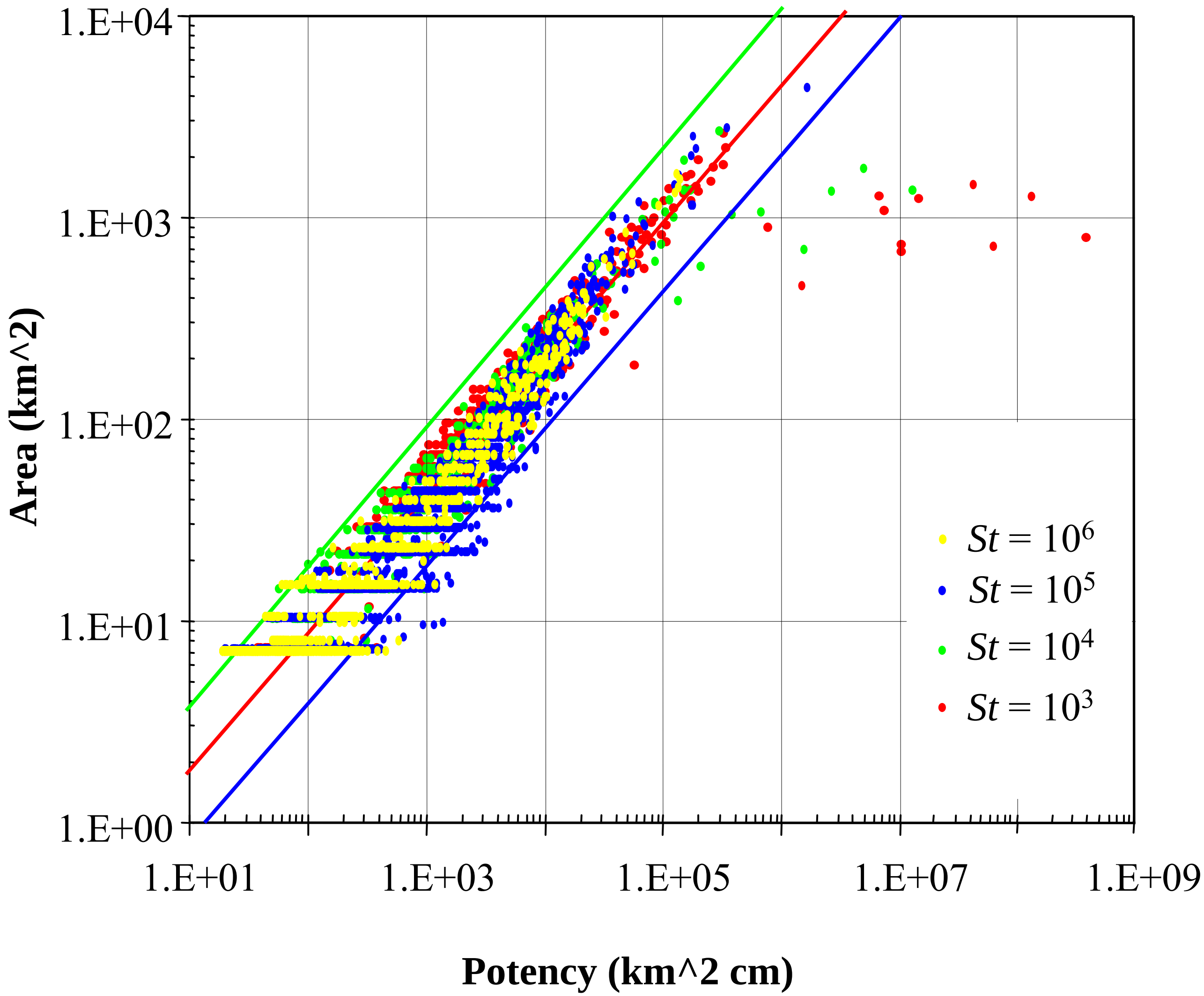


Damage decrease (healing) for different healing parameter C2 and 200 MPa lithostatic pressure



Simulated potency-area results for models with different stiffness *St* of the virtual boundary springs.

The other model parameters are w=0.1, C_1=10^{-10} s^{-1}, C_2=0.05



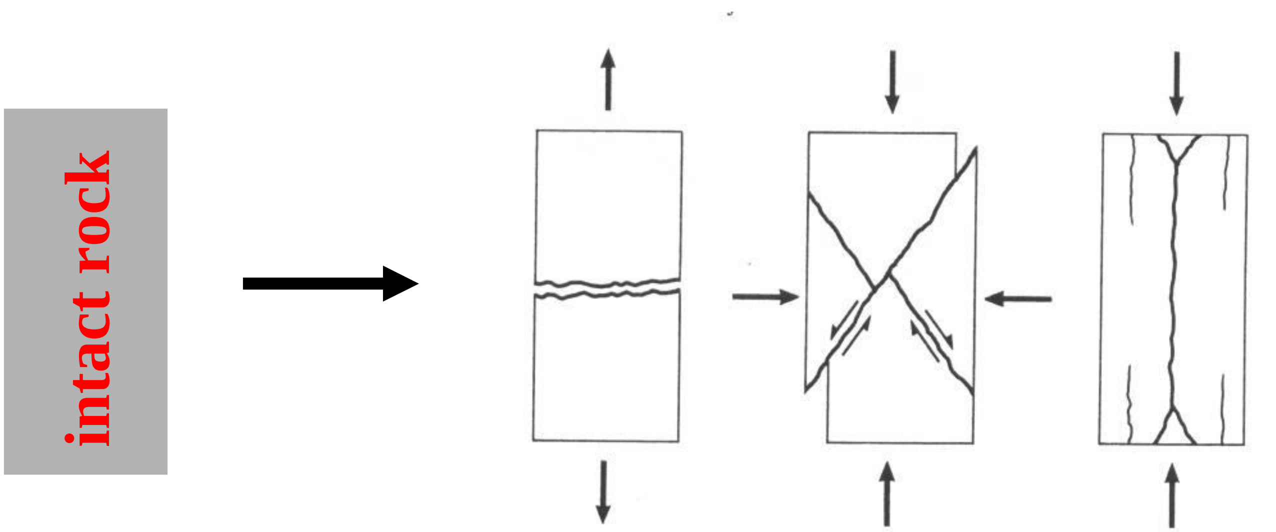
The presented damage rheology model reproduces the important observations of rock deformation

The model provides a framework for fault evolution and earthquakes

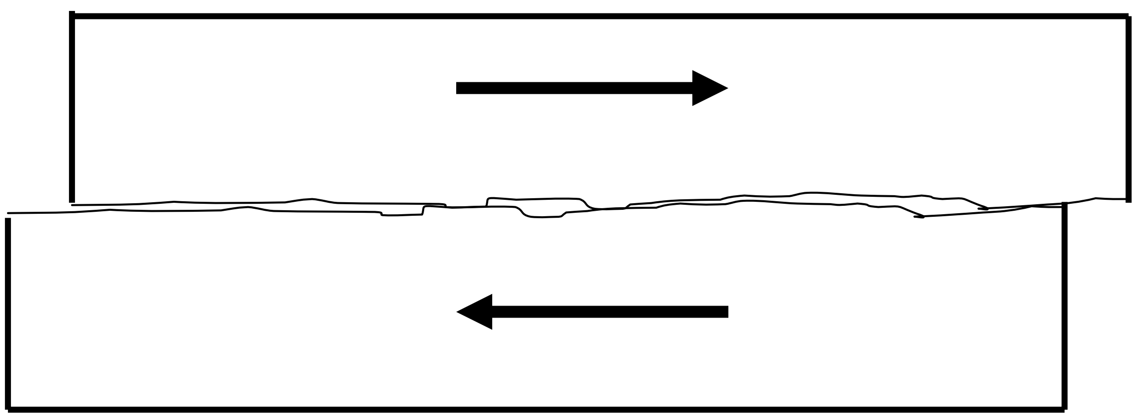
Thank you!

Brittle rock deformation is associated with

fracture (Coulomb failure stress)



friction - pre-existing fault

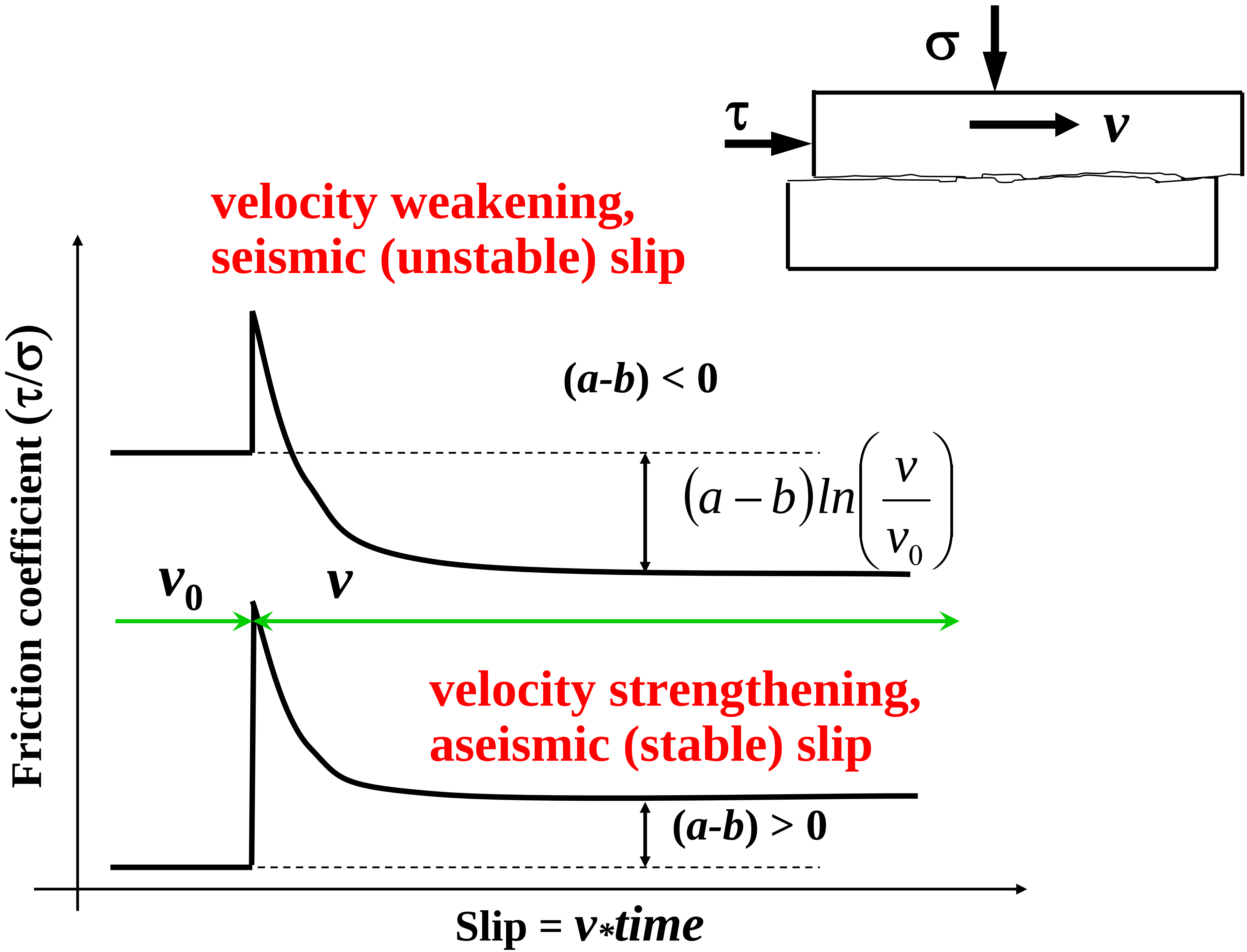


Friction models provide a conceptual framework incorporating the main stages of an earthquake cycle, and are widely used in seismology.

However…

- Deformation occurs only on well defined frictional surfaces

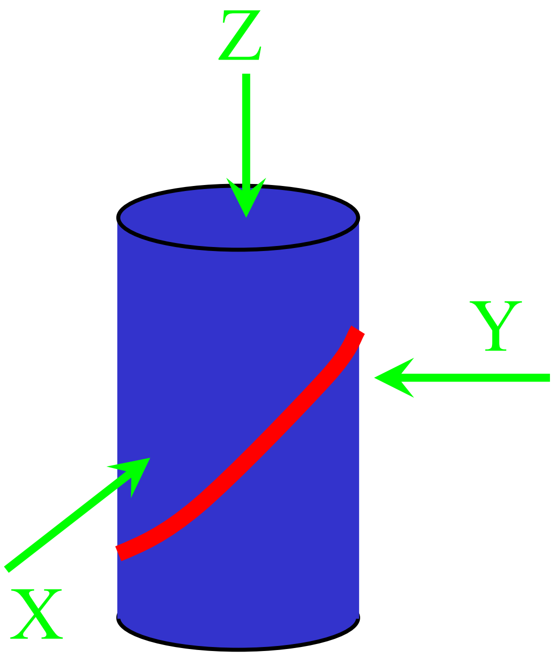
- It does not provide a mechanism for understanding:
  - Distributed deformation
  - Nucleation of new surfaces

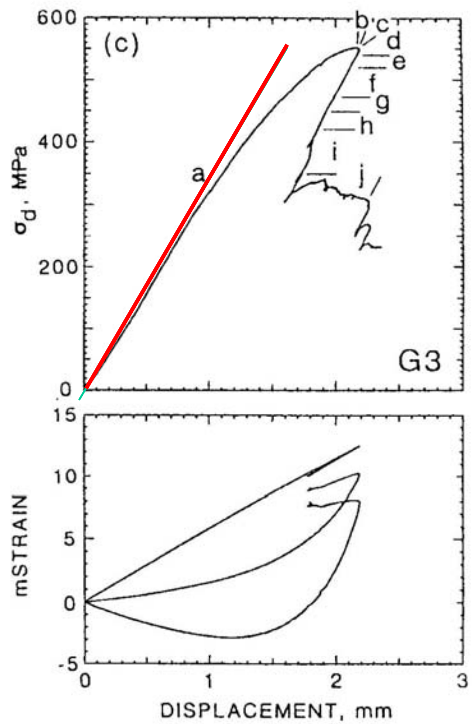


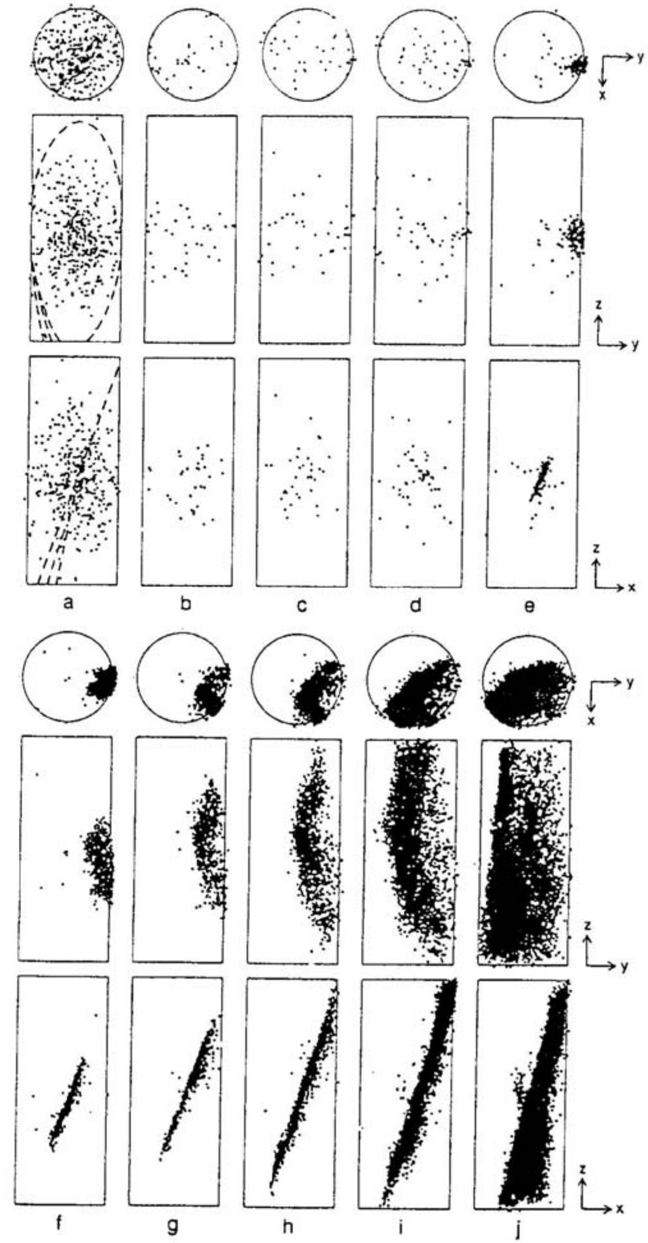
σ
τ
v
velocity weakening,
seismic (unstable) slip
(a-b) < 0
$(a-b)\,ln\left(\frac{v}{v_0}\right)$
v_0
v
velocity strengthening,
aseismic (stable) slip
(a-b) > 0
Friction coefficient (τ/σ)
Slip = v*time

Experimental observations with intact rock: from distributed damage to localized fracture zone

Stress-strains and AE locations for sample G3 from Lockner et al., 1992







Westerly granite,
50 MPa confining pressure

Why we need rheological model?

Brittle rock deformation is associated with

- **fracture**
- **friction**

Fracture is dominant in deformation of rock without a pre-existing macroscopic failure zone, Coulomb criteria and Mohr circle analysis

Friction is dominant in situations with existing sliding surfaces

HEALING

Motivated by the observed logarithmic increase of the static coefficient of friction (e.g., Dieterich, 1972, 1978)
a damage-dependent function for the kinetics of healing is

$$C(\alpha) = C_1 \cdot \exp\left(\frac{\alpha}{C_2}\right), \qquad \text{for } \frac{d\alpha}{dt} < 0$$

$$\alpha(t) = \alpha_0 - C_2 \cdot \ln\left(1 - \frac{C_1}{C_2} \cdot \exp\left[\frac{\alpha_0}{C_2}\right] \cdot \frac{\partial F}{\partial \alpha} \cdot t\right)$$

For constant loading (stationary contact)

Static friction: log-in-time increase

Time-Dependent Friction in Rock
Dieterich, JGR, 1972

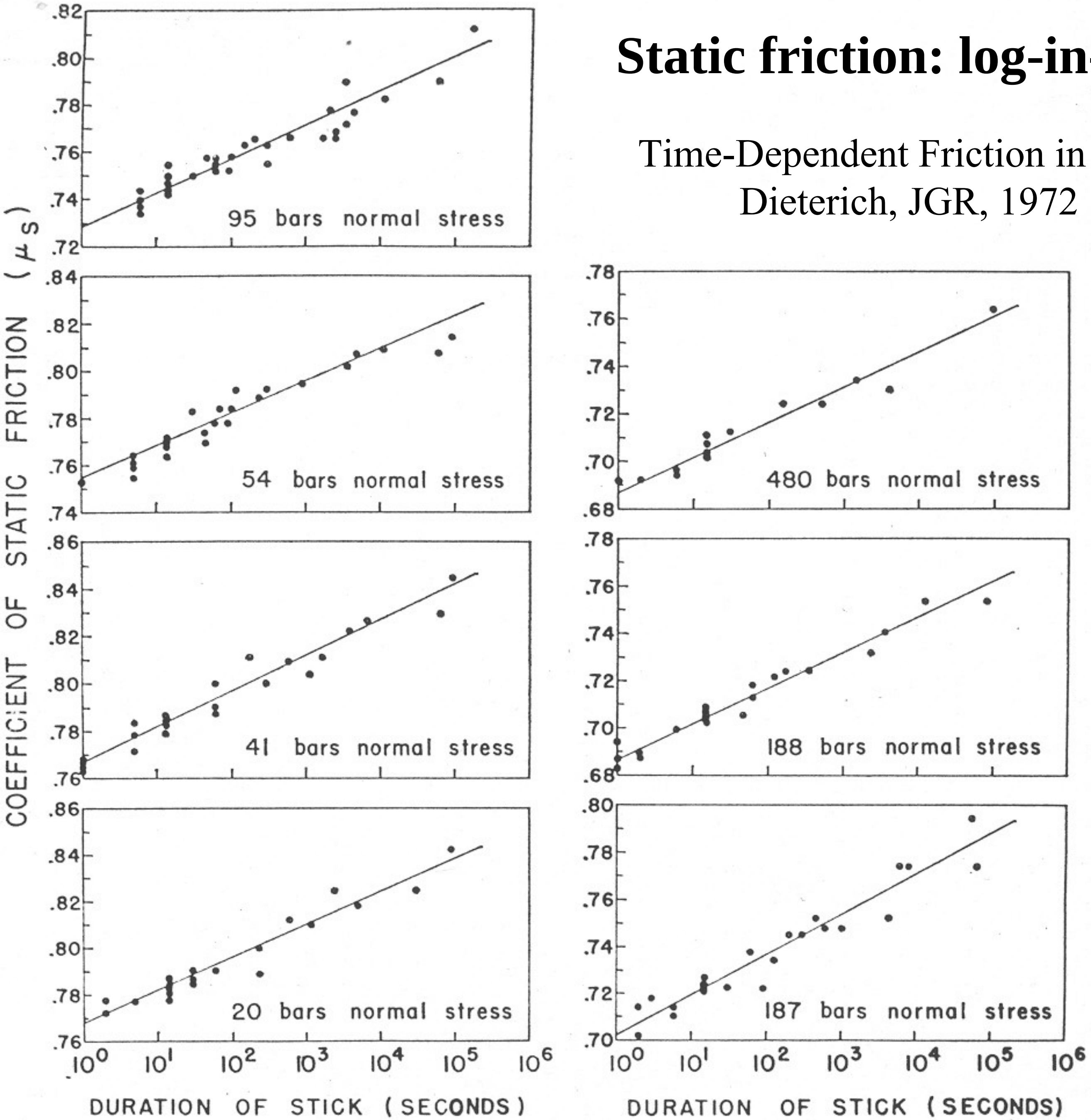
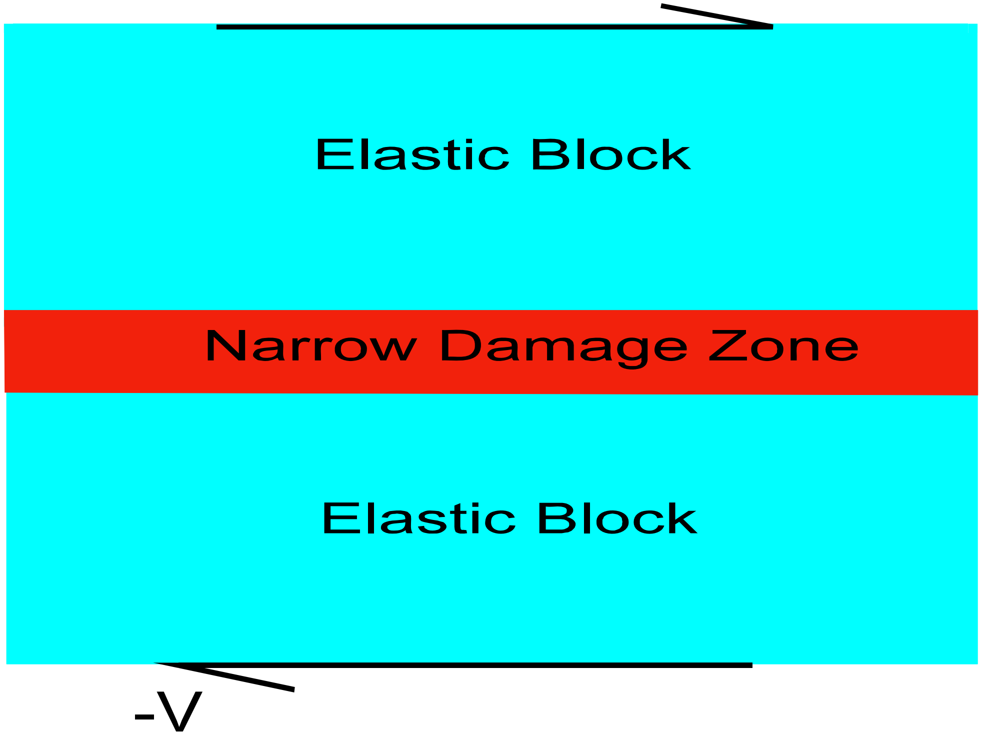


Fig. 2. Time dependence of the coefficient of static friction μ_s for sandstone.

$$f_s = f_0 + A \log_{10}(1+Bt)$$

$f_0 = 0.6\text{-}0.8$
$A = (1\text{-}3)\ 10^{-2}$
$B = (1\text{-}2)\ s^{-1}$

Friction



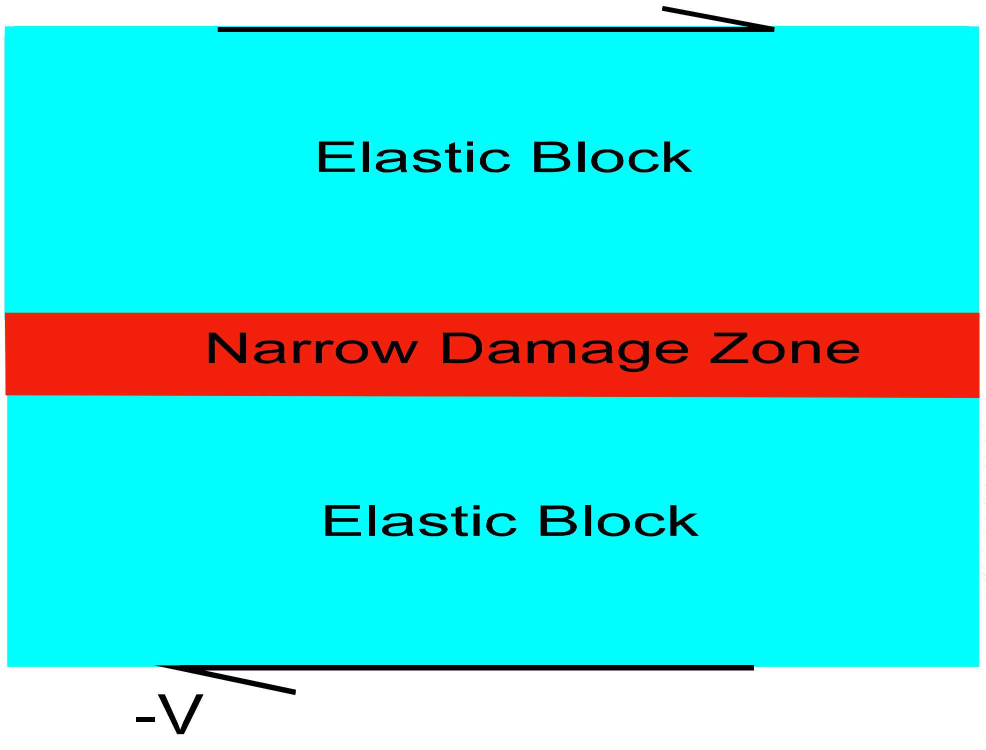
$$\varepsilon_e = \varepsilon_{cr}$$

$$\mu_\alpha = \frac{V\eta}{h\varepsilon_{cr}}$$

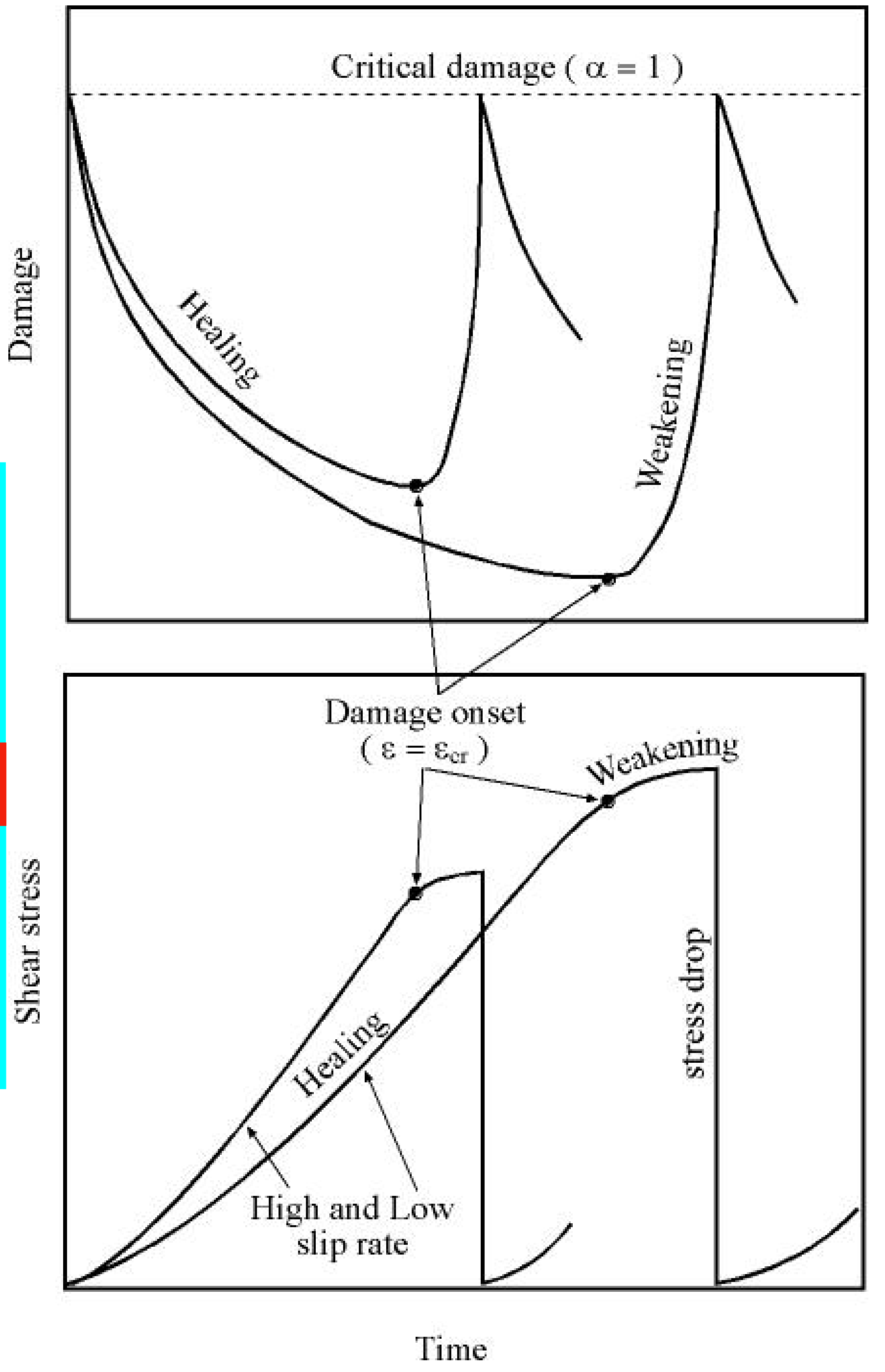
$$\frac{d\mu_\alpha}{dt} = -\mu_0 C(\alpha)\left[\varepsilon_e^2 - \varepsilon_{cr}^2\right]$$

$$\frac{d\varepsilon_e}{dt} = \frac{h\mu}{h\mu + L\mu_\alpha}\left[\frac{L}{h}\frac{\mu_0}{\mu}\varepsilon_e C(\alpha)\left(\varepsilon_e^2 - \varepsilon_{cr}^2\right) + \frac{V}{h} - \frac{\mu_\alpha \varepsilon_e}{\eta}\right]$$

Friction



Elastic Block
Narrow Damage Zone
Elastic Block
-V

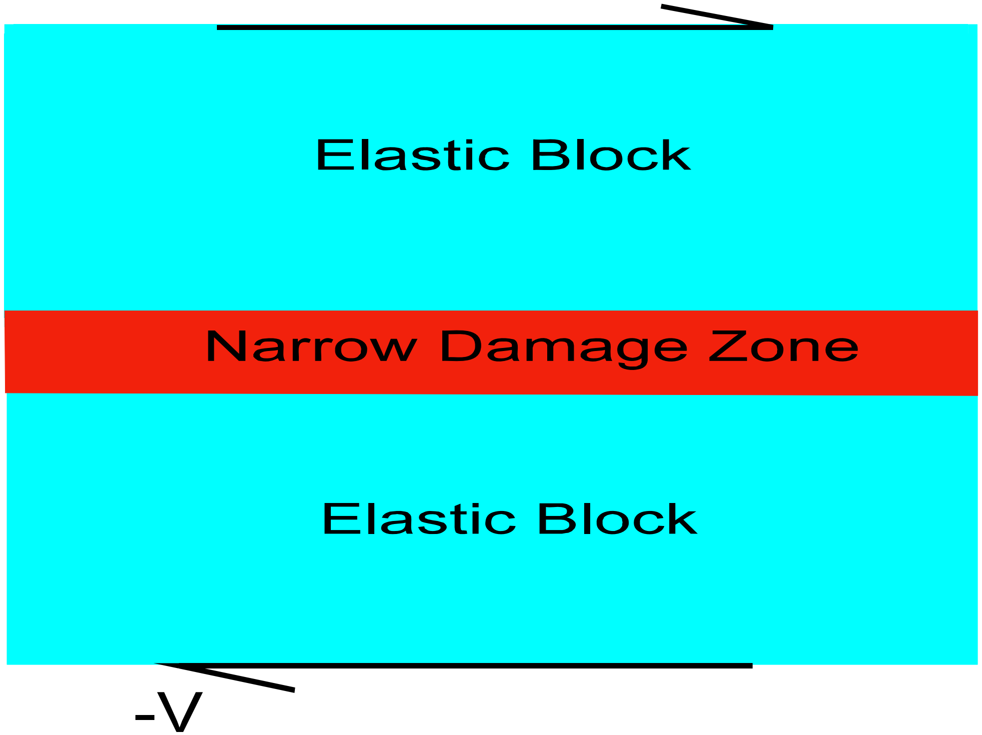


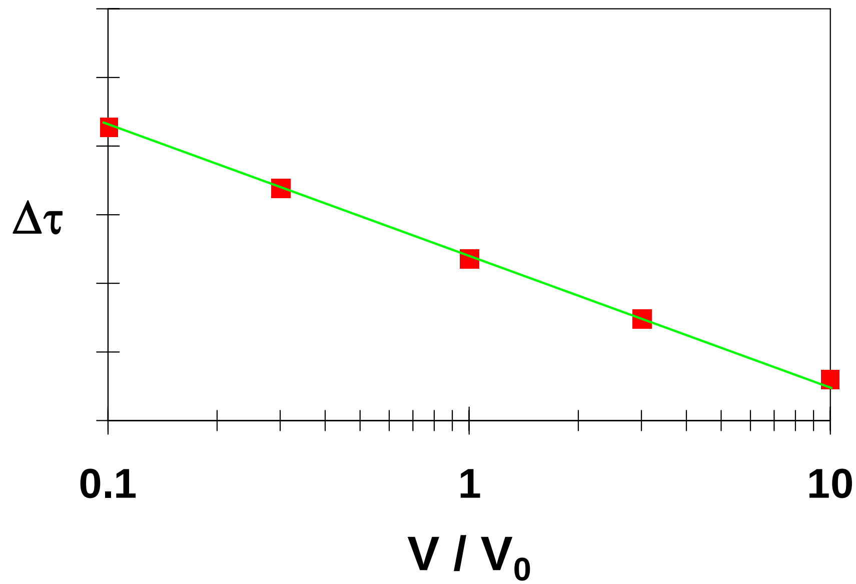
Critical damage (α = 1)
Damage
Healing
Weakening
Damage onset
(ε = εcr)
Weakening
Shear stress
Healing
stress drop
High and Low
slip rate
Time

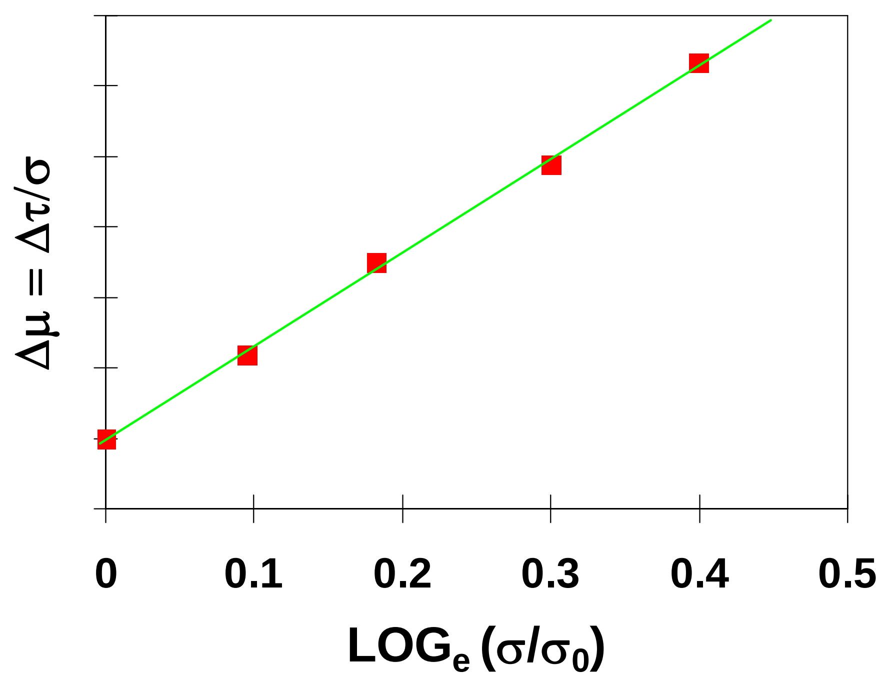
$$C_1 = -BC_2 \cdot exp\left[\frac{-\alpha_0}{C_2}\right] / \frac{\partial F}{\partial \alpha} \approx BC_2 \cdot exp\left[\frac{-\alpha_0}{C_2}\right] / \varepsilon_{CMP}^2$$

$$\tau_h = \frac{\mu_0 \varepsilon_{cr}}{2}\left[1 - C_2\, ln\left(V \frac{C_2}{hC_1}\right) + O(C_2)\right]$$

$$C_2 = -\frac{1}{\tau_h} \frac{\partial \tau_h}{\partial\, ln(V / V_0)} \sim -b$$

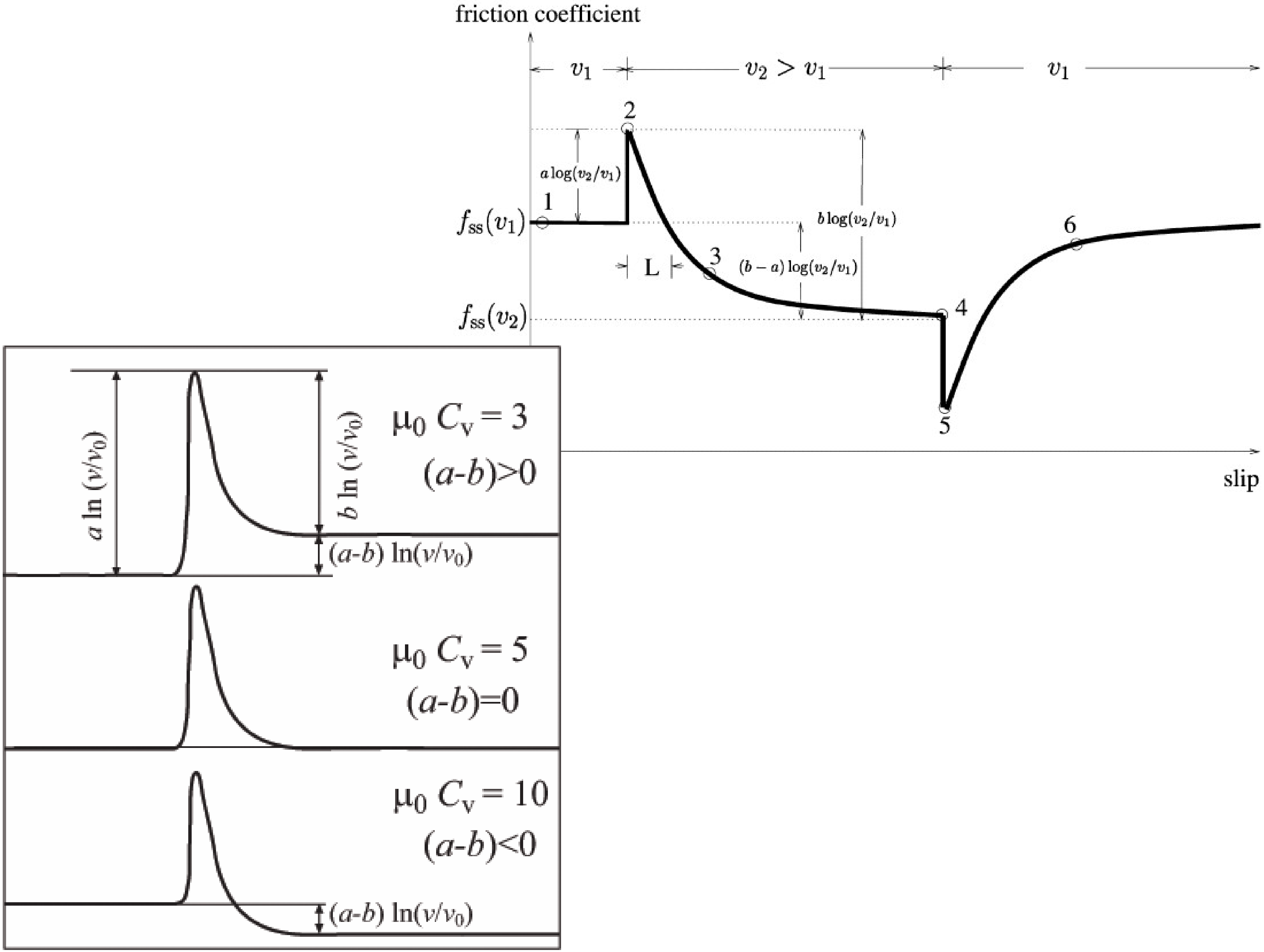






RESULT:

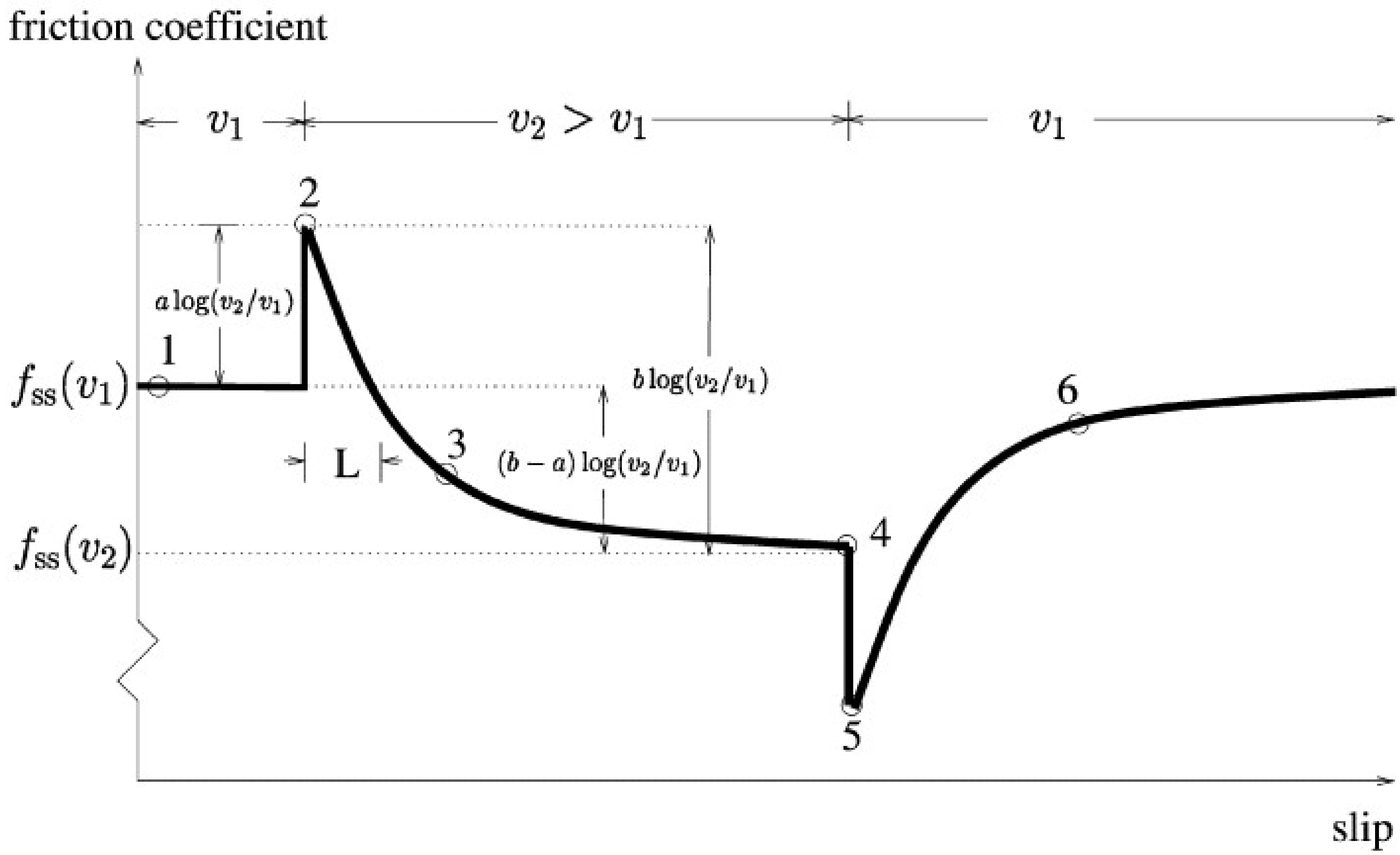
Rate- and state-dependent friction.


friction coefficient
v_1
$v_2 > v_1$
v_1
2
$a \log(v_2/v_1)$
1
$f_{ss}(v_1)$
$b \log(v_2/v_1)$
6
L
3
$(b-a)\log(v_2/v_1)$
$f_{ss}(v_2)$
4
5
slip
$a \ln(v/v_0)$
$b \ln(v/v_0)$
$\mu_0 C_v = 3$
$(a\text{-}b)>0$
$(a\text{-}b)\ln(v/v_0)$
$\mu_0 C_v = 5$
$(a\text{-}b)=0$
$\mu_0 C_v = 10$
$(a\text{-}b)<0$
$(a\text{-}b)\ln(v/v_0)$

The rate- and state-dependent friction model [*e.g.*, *Dieterich*, 1979, 1981; *Ruina*, 1983] provides a conceptual framework incorporating the main stages of an earthquake cycle, including stable and unstable slip and is widely used in seismology.

However

- Deformation occurs on well defined frictional surfaces
- It does not provide a mechanism for understanding distributed deformation
- Out of plane rupture propagation
- Nucleation of new surfaces
- Arrest of the propagating rupture



friction coefficient
v_1
$v_2 > v_1$
v_1
2
$a \log(v_2/v_1)$
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$f_{ss}(v_1)$
$b \log(v_2/v_1)$
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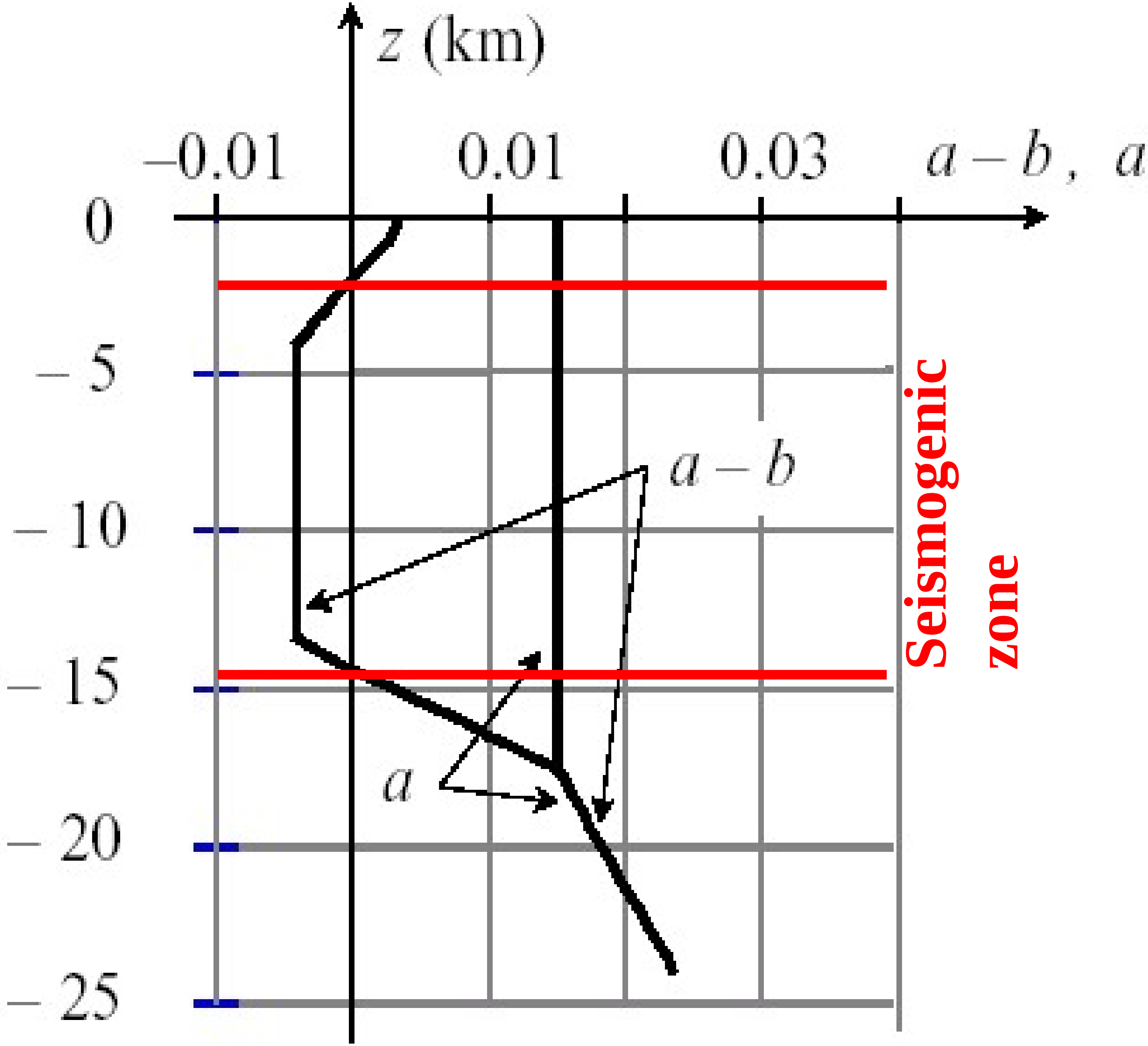
$(b-a)\log(v_2/v_1)$
$f_{ss}(v_2)$
4
5
slip



Ben-Zion and Rice, JGR 1993